

# Nil Bohr-sets and almost automorphy of higher order

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ABSTRACT. Two closely related topics: higher order Bohr sets and higher order almost automorphy are investigated in this paper. Both of them are related to nilsystems.

In the first part, the problem which can be viewed as the higher order version of an old question concerning Bohr sets is studied: for any  $d \in \mathbb{N}$  does the collection of  $\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$  with  $S$  syndetic coincide with that of  $\text{Nil}_d$  Bohr<sub>0</sub>-sets? It is proved that  $\text{Nil}_d$  Bohr<sub>0</sub>-sets could be characterized via generalized polynomials, and applying this result one side of the problem is answered affirmatively: for any  $\text{Nil}_d$  Bohr<sub>0</sub>-set  $A$ , there exists a syndetic set  $S$  such that  $A \supset \{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$ . Moreover, it is shown that the answer of the other side of the problem can be deduced from some result by Bergelson-Host-Kra if modulo a set with zero density.

In the second part, the notion of  $d$ -step almost automorphic systems with  $d \in \mathbb{N} \cup \{\infty\}$  is introduced and investigated, which is the generalization of the classical almost automorphic ones. It is worth to mention that some results concerning higher order Bohr sets will be applied to the investigation. For a minimal topological dynamical system  $(X, T)$  it is shown that the condition  $x \in X$  is  $d$ -step almost automorphic can be characterized via various subsets of  $\mathbb{Z}$  including the dual sets of  $d$ -step Poincaré and Birkhoff recurrence sets, and  $\text{Nil}_d$  Bohr<sub>0</sub>-sets. Moreover, it turns out that the condition  $(x, y) \in X \times X$  is regionally proximal of order  $d$  can also be characterized via various subsets of  $\mathbb{Z}$ .

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## CHAPTER 1

### Introduction

In this paper we study two closely related topics: higher order Bohr sets and higher order almost automorphy. Both of them are related to nilsystems. In the first part we investigate the higher order Bohr sets. Then in the second part we study the higher order automorphy, and explain how these two topics are closely related.

#### 1.1. Higher order Bohr problem

A very old problem from combinatorial number theory and harmonic analysis, rooted in the classical work of Bogoliuboff, Følner [17], Ellis-Keynes [16], and Veech [52] is the following. Let  $S$  be a syndetic subset of the integers. Is the set  $S - S$  a Bohr neighborhood of zero in  $\mathbb{Z}$  (also called Bohr<sub>0</sub>-set)? For the equivalent statements and results related to the problem in combinatorial number theory, group theory and dynamical systems, see Glasner [26], Weiss [54], Katznelson [42], Pestov [47], Boshernitzan-Glasner [11], Huang-Ye [41], Grivaux-Roginskaya [32, 33].

Bohr-sets are fundamentally abelian in nature. Nowadays it has become apparent that higher order non-abelian Fourier analysis plays an important role both in combinatorial number theory and ergodic theory. Related to this, a higher-order version of Bohr<sub>0</sub> sets, namely Nil<sub>d</sub> Bohr<sub>0</sub>-sets, was introduced in [35]. For the recent results obtained by Bergelson-Furstenberg-Weiss and Host-Kra, see [4, 35].

**1.1.1. Nil Bohr-sets.** There are several equivalent definitions for Bohr-sets. Here is the one easy to understand: a subset  $A$  of  $\mathbb{Z}$  is a *Bohr-set* if there exist  $m \in \mathbb{N}$ ,  $\alpha \in \mathbb{T}^m$ , and a non-empty open set  $U \subset \mathbb{T}^m$  such that  $\{n \in \mathbb{Z} : n\alpha \in U\}$  is contained in  $A$ ; the set  $A$  is a *Bohr<sub>0</sub>-set* if additionally  $0 \in U$ .

It is not hard to see that if  $(X, T)$  is a minimal equicontinuous system,  $x \in X$  and  $U$  is a neighborhood of  $x$ , then  $N(x, U) =: \{n \in \mathbb{Z} : T^n x \in U\}$  contains  $S - S =: \{a - b : a, b \in S\}$  with  $S$  syndetic, i.e. with a bounded gap ( $S$  can be chosen as  $N(x, U_1)$ , where  $U_1 \subset U$  is an open neighborhood of  $x$ ). This implies that if  $A$  is a Bohr<sub>0</sub>-set, then  $A \supset S - S$  with  $S$  syndetic. The old question concerning Bohr<sub>0</sub>-sets is

**Problem A-I:** Let  $S$  be a syndetic subset of  $\mathbb{Z}$ , is  $S - S$  a Bohr<sub>0</sub>-set?

Note that Ellis-Keynes [16] proved that  $S - S + S - a$  is a Bohr<sub>0</sub>-set for some  $a \in S$ . Veech showed that it is at least “almost” true [52]. That is, given a syndetic set  $S \subset \mathbb{Z}$ , there is an  $N \subset \mathbb{Z}$  with density zero such that  $(S - S)\Delta N$  is a Bohr<sub>0</sub>-set. Kříž [44] showed that there exists a subset  $K$  of  $\mathbb{Z}$  with positive upper Banach density such that  $K - K$  does not contains  $S - S$  for any syndetic subset  $S$  of  $\mathbb{Z}$ .

This implies that Problem A-I has a negative answer if we replace a syndetic subset of  $\mathbb{Z}$  by a subset of  $\mathbb{Z}$  with positive upper Banach density.

A subset  $A$  of  $\mathbb{Z}$  is a  $Nil_d$  Bohr<sub>0</sub>-set if there exist a  $d$ -step nilsystem  $(X, T)$ ,  $x_0 \in X$  and an open neighborhood  $U$  of  $x_0$  such that  $N(x_0, U) =: \{n \in \mathbb{Z} : T^n x_0 \in U\}$  is contained in  $A$ . Denote by  $\mathcal{F}_{d,0}$  the family<sup>1</sup> consisting of all  $Nil_d$  Bohr<sub>0</sub>-sets. We can now formulate a higher order form of Problem A-I. We note that

$$\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$$

can be viewed as the common differences of arithmetic progressions with length  $d+1$  appeared in the subset  $S$ . In fact,  $S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset$  if and only if there is  $m \in S$  with

$$m, m + n, \dots, m + dn \in S.$$

Particularly,  $S - S = \{n \in \mathbb{Z} : S \cap (S - n) \neq \emptyset\}$ .

**Problem B-I:** [Higher order form of Problem A-I] Let  $d \in \mathbb{N}$ .

- (1) For any  $Nil_d$  Bohr<sub>0</sub>-set  $A$ , is it true that there is a syndetic subset  $S$  of  $\mathbb{Z}$  with  $A \supset \{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$ ?
- (2) For any syndetic subset  $S$  of  $\mathbb{Z}$ , is  $\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$  a  $Nil_d$  Bohr<sub>0</sub>-set?

**1.1.2. Dynamical version of the higher order Bohr problem.** Sometimes combinatorial questions can be translated into dynamical ones by the Furstenberg correspondence principle, see Section 2.3.1. Using this principle, it can be shown that Problem A-I is equivalent to the following version:

**Problem A-II:** For any minimal system  $(X, T)$  and any nonempty open subset  $U$  of  $X$ , is the set  $\{n \in \mathbb{Z} : U \cap T^{-n}U \neq \emptyset\}$  a Bohr<sub>0</sub>-set?

Similarly, Problem B-I has its dynamical version:

**Problem B-II:** [Dynamical version of Problem B-I] Let  $d \in \mathbb{N}$ .

- (1) For any  $Nil_d$  Bohr<sub>0</sub>-set  $A$ , is it true that there are a minimal system  $(X, T)$  and a non-empty open subset  $U$  of  $X$  with

$$A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}?$$

- (2) For any minimal system  $(X, T)$  and any non-empty open subset  $U$  of  $X$ , is it true that  $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$  is a  $Nil_d$  Bohr<sub>0</sub>-set?

In the next section, we will give the third version of Problem B via recurrence sets. The equivalence of three versions will be shown in Chapter 2.

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<sup>1</sup> A collection  $\mathcal{F}$  of subsets of  $\mathbb{Z}$  (or  $\mathbb{N}$ ) is a *family* if it is hereditary upward, i.e.  $F_1 \subset F_2$  and  $F_1 \in \mathcal{F}$  imply  $F_2 \in \mathcal{F}$ . Any nonempty collection  $\mathcal{A}$  of subsets of  $\mathbb{Z}$  generates a family  $\mathcal{F}(\mathcal{A}) := \{F \subset \mathbb{Z} : F \supset A \text{ for some } A \in \mathcal{A}\}$ .

**1.1.3. Main results on the higher order Bohr problem.** We aim to study Problem B-I or its dynamical version Problem B-II. We will show that Problem B-II(1) has an affirmative answer, and Problem B-II(2) has a positive answer if ignoring a set with zero density. Namely, we will show

**Theorem A:** *Let  $d \in \mathbb{N}$ .*

- (1) *If  $A \subset \mathbb{Z}$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set, then there exist a minimal  $d$ -step nilsystem  $(X, T)$  and a nonempty open subset  $U$  of  $X$  with*

$$A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

- (2) *For any minimal system  $(X, T)$  and any non-empty open subset  $U$  of  $X$ ,  $I = \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$  is almost a  $\text{Nil}_d$  Bohr<sub>0</sub>-set, i.e. there is  $M \subset \mathbb{Z}$  with zero upper Banach density such that  $I \Delta M$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set*

As we said before for  $d = 1$  Theorem A(1) can be easily proved. To show Theorem A(1) in the general case, we need to investigate the properties of  $\mathcal{F}_{d,0}$ . It is interesting that in the process to do this, generalized polynomials (see §4.1 for a definition) appear naturally. Generalized polynomials have been studied extensively, see for example the remarkable paper by Bergelson and Leibman [6] and references therein. After finishing this paper we even find that it also plays an important role in the recent work by Green, Tao and Ziegler [31]. In fact the special generalized polynomials defined in this paper are closely related to the nilcharacters defined there. We remark that Theorem A(2) was first proved by Veech in the case  $d = 1$  [52], and its proof will be presented in Section 2.2.

Let  $\mathcal{F}_{GP_d}$  (resp.  $\mathcal{F}_{SGP_d}$ ) be the family generated by the sets of forms

$$\bigcap_{i=1}^k \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\},$$

where  $k \in \mathbb{N}$ ,  $P_1, \dots, P_k$  are generalized polynomials (resp. special generalized polynomials) of degree  $\leq d$ , and  $\epsilon_i > 0$ . For the precise definitions see Chapter 4. We remark that one can in fact show that  $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$  (Theorem 4.2.11).

The following theorem illustrates the relation between  $\text{Nil}_d$  Bohr<sub>0</sub>-sets and the sets defined above using generalized polynomials.

**Theorem B:** *Let  $d \in \mathbb{N}$ . Then  $\mathcal{F}_{d,0} = \mathcal{F}_{GP_d}$ .*

When  $d = 1$ , we have  $\mathcal{F}_{1,0} = \mathcal{F}_{SGP_1}$ . This is the result of Katznelson [42], since  $\mathcal{F}_{SGP_1}$  is generated by sets of forms  $\bigcap_{i=1}^k \{n \in \mathbb{Z} : na_i \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\}$  with  $k \in \mathbb{N}$ ,  $a_i \in \mathbb{R}$  and  $\epsilon_i > 0$ .

Theorem A(1) follows from Theorem B and the following result:

**Theorem C:** *Let  $d \in \mathbb{N}$ . If  $A \in \mathcal{F}_{GP_d}$ , then there exist a minimal  $d$ -step nilsystem  $(X, T)$  and a nonempty open subset  $U$  of  $X$  such that*

$$A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

The proof of Theorem B is divided into two parts, namely

**Theorem B(1):**  $\mathcal{F}_{d,0} \subset \mathcal{F}_{GP_d}$  and

**Theorem B(2):**  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ .

The proof of Theorem B(1) is a theoretical argument using nilpotent Lie group theory; and the proofs of Theorem B(2) and Theorem C involve very complicated construction and computation where nilpotent matrix Lie groups are used.

REMARK 1.1.1. Our definition of generalized polynomials is slight different from the ones defined in [6]. In fact we need to specialize the degree of the generalized polynomials which is not needed in [6]. Moreover, our Theorem B can be compared with Theorem A of Bergelson and Leibman proved in [6].

## 1.2. Higher order almost automorphy

The notion of almost automorphy was first introduced by Bochner in [9, 10], and Veech studied almost automorphic systems in [51] using Fourier analysis. To study higher order almost automorphic systems it is expected that nilpotent Lie groups and higher order Fourier analysis will be involved and it turns out that it is the case. We will apply results obtained in the first part to study higher order almost automorphic systems, namely  $d$ -step almost automorphic systems which by the definition are the almost one-to-one extensions of their maximal  $d$ -step nilfactors with  $d \in \mathbb{N} \cup \{\infty\}$  (an  $\infty$ -step nilsystem was defined in [13]). Since for a minimal system the maximal  $d$ -step nilfactor is induced by the regionally proximal relation of order  $d$  (which is a closed invariant equivalence relation [36, 48]), the natural way we study  $d$ -step almost automorphic systems is that we first show some characterizations of regionally proximal relation of order  $d$ , and then use them to obtain results for  $d$ -step almost automorphic systems. In the process of doing above many interesting subsets of  $\mathbb{Z}$  including higher order Poincaré and Birkhoff recurrence sets (usual and cubic versions), higher order Bohr-sets,  $SG_d$  sets (introduced in [35]) and others are involved. In this section we introduce some backgrounds of our study and then state the main results on higher order almost automorphy.

First we give some backgrounds.

### 1.2.1. Almost periodicity, almost automorphy and characterizations.

The study of (uniformly) almost periodic functions was initiated by Bohr in a series of three papers 1924-26 which can be found in [8]. The literature on almost periodic functions is enormous, and the notion has been generalized in several directions. Nowadays the theory of almost periodic functions may be recognized as the representation theory of compact Hausdorff groups: every topological group  $G$  has a group compactification  $\alpha_G : G \rightarrow bG$  such that the space of almost periodic functions on  $G$  is just the set of all functions  $f \circ \alpha_G$  with  $f \in C(bG)$ . The compactification  $(\alpha_G, bG)$  of  $G$  is called the *Bohr compactification* of  $G$ .

A class of functions related to the almost periodic ones is the class of *almost automorphic functions*: these functions turn out to be the ones of the form  $h \circ \alpha_G$  with  $h$  a bounded continuous function on  $\alpha_G(G)$  (if  $h$  is uniformly continuous and



bounded on  $\alpha_G(G)$ , then it extends to an  $f \in C(bG)$ , so  $h \circ \alpha_G = f \circ \alpha_G$  is almost periodic on  $G$ ).

The notion of almost automorphy was first introduced by Bochner in 1955 in a work of differential geometry [9, 10]. Taking  $G$  for the present to be the group of integers  $\mathbb{Z}$ , an almost automorphic function  $f$  has the property that from any sequence  $\{n'_i\} \subset \mathbb{Z}$  one may extract a subsequence  $\{n_i\}$  such that both

$$\lim_{i \rightarrow \infty} f(t + n_i) = g(t) \quad \text{and} \quad \lim_{i \rightarrow \infty} g(t - n_i) = f(t)$$

hold for each  $t \in \mathbb{Z}$  and some function  $g$ , not necessarily uniformly. Bochner [10] has observed that almost periodic functions are almost automorphic, but the converse is not true. Veech [51] showed that the almost automorphic functions can be characterized in terms of the almost periodic ones, and vice versa. In the same paper, Veech considered the system associated with an almost automorphic function, and introduced the notion of *almost automorphic point* (AA point, for short) in topological dynamical systems (t.d.s. for short). For a t.d.s.  $(X, T)$ , a point  $x \in X$  is said to be *almost automorphic* if from any sequence  $\{n'_i\} \subset \mathbb{Z}$  one may extract a subsequence  $\{n_i\}$  such that

$$\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} T^{n_i - n_j} x = x.$$

Moreover, Veech [51, 52] gave the structure theorem for minimal systems with an AA point: each minimal AA system is an almost one-to-one extension of its maximal equicontinuous factor.

The notion of almost automorphy is very useful in the study of differential equations, and see [49] and references therein for more information on this topic.

To state other characterizations of an AA point we need to introduce the notion of regionally proximal relations, and Poincaré and Birkhoff recurrence sets.

Let us first discuss regionally proximal relations. For a t.d.s.  $(X, T)$ , it was proved in [15] that there exists a closed  $T$ -invariant equivalence relation  $S_{eq}$  on  $X$  such that  $(X/S_{eq}, T)$  is the maximal equicontinuous factor.  $S_{eq}$  is called the *equicontinuous structure relation*. It was also showed in [15] that  $S_{eq}$  is the smallest closed  $T$ -invariant equivalence relation containing the regionally proximal relation  $\mathbf{RP} = \mathbf{RP}(X, T)$  (recall that  $(x, y) \in \mathbf{RP}$  if there are sequences  $x_i, y_i \in X, n_i \in \mathbb{Z}$  such that  $x_i \rightarrow x, y_i \rightarrow y$  and  $(T \times T)^{n_i}(x_i, y_i) \rightarrow (z, z), i \rightarrow \infty$ , for some  $z \in X$ ). A natural question was whether  $S_{eq} = \mathbf{RP}(X)$  for all minimal t.d.s.? Veech [52] gave the first positive answer to this question, i.e. he proved that  $S_{eq} = \mathbf{RP}(X)$  for all minimal t.d.s. under abelian group actions. As a matter of fact, Veech proved that for a minimal t.d.s.  $(x, y) \in S_{eq}$  if and only if there is a sequence  $\{n_i\} \subset \mathbb{Z}$  and  $z \in X$  such that

$$T^{n_i} x \rightarrow z \quad \text{and} \quad T^{-n_i} z \rightarrow y, \quad i \rightarrow \infty.$$

As a direct corollary, for a minimal t.d.s.  $(X, T)$ , a point  $x \in X$  is AA if and only if

$$\mathbf{RP}[x] = \{y \in X : (x, y) \in \mathbf{RP}\} = \{x\}.$$

Also from Veech's approach, it is easy to show that for a minimal t.d.s.  $(X, T)$ ,  $(x, y) \in \mathbf{RP}$  if and only if for each neighborhood  $U$  of  $y$ ,  $N(x, U) =: \{n \in \mathbb{Z} : T^n x \in U\}$

$U\}$  contains some  $\Delta$ -set<sup>2</sup>. Hence one can obtain an equivalent condition for an AA point [21, Theorem 9.13]: *a point  $x \in X$  is AA if and only if it is  $\Delta^*$ -recurrent*<sup>3</sup> (this result will be reproved by a different method as a special case of our theorems). For other properties related to  $\Delta^*$ -sets, see [4, 35].

Now we discuss Poincaré and Birkhoff recurrence sets. The Birkhoff recurrence theorem states that each t.d.s. has a recurrent point which implies that whenever  $(X, T)$  is a minimal t.d.s. and  $U \subset X$  a nonempty open set,  $N(U, U) =: \{n \in \mathbb{Z} : U \cap T^{-n}U \neq \emptyset\}$  is infinite. The measurable version of this phenomenon is the well known Poincaré's recurrence theorem: let  $(X, \mathcal{X}, \mu, T)$  be a measure preserving system and  $A \in \mathcal{X}$  with  $\mu(A) > 0$ , then  $N_\mu(A, A) =: \{n \in \mathbb{Z} : \mu(A \cap T^{-n}A) > 0\}$  is infinite.

In [21, 20] Furstenberg introduced the notion of Poincaré and Birkhoff recurrence sets. A subset  $P$  of  $\mathbb{Z}$  is called a *Poincaré recurrence set* if whenever  $(X, \mathcal{X}, \mu, T)$  is a measure preserving system and  $A \in \mathcal{X}$  has positive measure, then  $P \cap N_\mu(A, A) \neq \emptyset$ . Similarly, a subset  $P$  of  $\mathbb{Z}$  is called a *Birkhoff recurrence set* if whenever  $(X, T)$  is a minimal t.d.s. and  $U \subset X$  a nonempty open set,  $P \cap N(U, U) \neq \emptyset$ . Let  $\mathcal{F}_{Poi}$  and  $\mathcal{F}_{Bir}$  denote the collections of Poincaré and Birkhoff recurrence sets of  $\mathbb{Z}$  respectively.

In [40], it was shown that for a minimal t.d.s.  $(x, y) \in \mathbf{RP}$  if and only if for each neighborhood  $U$  of  $y$ ,  $N(x, U) \in \mathcal{F}_{Poi}$ . We will show that one can use  $\mathcal{F}_{Poi}$  to get another equivalent condition for an AA point: *a point  $x \in X$  is AA if and only if it is  $\mathcal{F}_{Poi}^*$ -recurrent*, where  $\mathcal{F}_{Poi}^*$  is the collection of subsets of  $\mathbb{Z}$  intersecting all sets from  $\mathcal{F}_{Poi}$ . One has similar results for Birkhoff recurrence sets.

**1.2.2. Nilfactors and higher order almost automorphy.** In the 1970's Furstenberg gave a beautiful proof of Szemerédi's theorem via ergodic theory [19]. It remains a question if the multiple ergodic averages  $\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \dots f_d(T^{dn} x)$  converges in  $L^2(X, \mu)$  for  $f_1, \dots, f_d \in L^\infty(X, \mu)$ . This question was finally answered by Host and Kra in [34] (see also Ziegler in [56]).

The authors in [34] defined for each  $d \in \mathbb{N}$  and each measure-preserving transformation on the probability space  $(X, \mathcal{B}, \mu)$  a factor  $\mathcal{Z}_d$  which is characteristic and is an inverse limit of  $d$ -step nilsystems. Since topological dynamics and ergodic theory are 'twins', it is natural to ask how to obtain similar factors in topological dynamics. In the pioneer paper [36] Host-Kra-Maass succeeded doing the job for minimal distal systems. Namely, for each  $d \in \mathbb{N}$  and each t.d.s.  $(X, T)$  they defined  $\mathbf{RP}^{[d]}(X, T)$  (the regionally proximal relation of order  $d$ ) and showed that  $\mathbf{RP}^{[d]}(X, T)$  is an equivalence relation when  $(X, T)$  is minimal distal, and  $X/\mathbf{RP}^{[d]}(X, T)$  is the maximal  $d$ -step nilfactor of  $(X, T)$ . Recently, Shao and Ye [48] proved that the above conclusion holds for general minimal systems. We note that the counterpart of the

<sup>2</sup>A  $\Delta$ -set is obtained by taking an arbitrary sequence in  $\mathbb{Z}$ ,  $\{s_n\}$  and forming the difference  $\{s_n - s_m : n > m\}$ . A  $\Delta^*$  set is a subset of  $\mathbb{Z}$  intersecting all  $\Delta$ -sets.

<sup>3</sup>Let  $\mathcal{F}$  be a collection of subsets of  $\mathbb{Z}$  and let  $(X, T)$  be a t.d.s. A point  $x$  of  $X$  is called  $\mathcal{F}$ -recurrent if  $N(x, U) \in \mathcal{F}$  for every neighborhood  $U$  of  $x$ .

characteristic factors in topological dynamics was also studied by Glasner [24, 25]. For the further study and applications of the factors, see [13, 37].

As we said before the notion of the regionally proximal relation of order  $d$  plays an important role in obtaining the maximal  $d$ -step nilfactors, see Section 2 for the definition. It is easy to see that  $\mathbf{RP}^{[d]}(X, T)$  is a closed and invariant relation for all  $d \in \mathbb{N}$ . When  $d = 1$ ,  $\mathbf{RP}^{[d]}(X, T)$  is nothing but the classical regionally proximal relation. Similar to the definition of almost automorphy, now we give the definition of  $d$ -step almost automorphy for all  $d \in \mathbb{N}$ . Let  $(X, T)$  be a minimal t.d.s. and  $d \in \mathbb{N}$ . A point  $x \in X$  is called  *$d$ -step almost automorphic* (or  *$d$ -step AA* for short) if  $\mathbf{RP}^{[d]}[x] = \{x\}$ . A minimal t.d.s. is called  *$d$ -step almost automorphic* if it has a  $d$ -step AA point. Since  $\mathbf{RP}^{[d]}$  is an equivalence relation for minimal t.d.s. [48], by definition it follows that for a minimal system  $(X, T)$ , it is a  $d$ -step AA system for some  $d \in \mathbb{N}$  if and only if it is an almost one-to-one extension of its maximal  $d$ -step nilfactor.

**1.2.3. Higher order recurrence sets.** In this paper we will use higher order recurrence sets to characterize  $d$ -step almost automorphy. To define them we need to state the multiple Poincaré and Birkhoff recurrence theorems, see [21].

- Let  $(X, \mathcal{X}, \mu, T)$  be a measure preserving system and  $d \in \mathbb{N}$ . Then for any  $A \in \mathcal{X}$  with  $\mu(A) > 0$  there is  $n \in \mathbb{N}$  such that  $\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0$ .
- Let  $(X, T)$  be a t.d.s. and  $d > 0$ . Then there are  $x \in X$  and a sequence  $\{n_i\}$  with  $n_i \rightarrow +\infty$  such that  $\lim_{i \rightarrow +\infty} T^{jn_i}x = x$  for each  $1 \leq j \leq d$ .

The facts enable us to get generalizations of Poincaré and Birkhoff recurrence sets (see [18]). Let  $d \in \mathbb{N}$ .

- (1) We say that  $S \subset \mathbb{Z}$  is a set of  *$d$ -recurrence* if for every measure preserving system  $(X, \mathcal{X}, \mu, T)$  and for every  $A \in \mathcal{X}$  with  $\mu(A) > 0$ , there exists  $n \in S$  such that  $\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0$ .
- (2) We say that  $S \subset \mathbb{Z}$  is a set of  *$d$ -topological recurrence* if for every minimal t.d.s.  $(X, T)$  and for every nonempty open subset  $U$  of  $X$ , there exists  $n \in S$  such that  $U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset$ .

**REMARK 1.2.1.** We remark that in (1) we can require that  $(X, \mathcal{X}, \mu, T)$  is ergodic (see the proof of Theorem 7.2.7). The above definitions are slightly different from the ones introduced in [18], namely we do not require  $n \neq 0$ . The main reason we define in this way is that for each  $A \in \mathcal{F}_{d,0}$ ,  $0 \in A$ . Thus  $\{0\} \cup C \in \mathcal{F}_{d,0}^*$  for each  $C \subset \mathbb{Z}$ , where  $\mathcal{F}_{d,0}^*$  is the dual family of  $\mathcal{F}_{d,0}$ , i.e. the collection of sets intersecting every  $\text{Nil}_d$  Bohr<sub>0</sub>-set.

Let  $\mathcal{F}_{\text{Poi}_d}$  (resp.  $\mathcal{F}_{\text{Bir}_d}$ ) be the family consisting of all sets of  $d$ -recurrence (resp. sets of  $d$ -topological recurrence). It is obvious by the above definition that  $\mathcal{F}_{\text{Poi}_d} \subset \mathcal{F}_{\text{Bir}_d}$ . Moreover, it is known that for each  $d \in \mathbb{N}$ ,  $\mathcal{F}_{\text{Poi}_d} \subsetneq \mathcal{F}_{\text{Poi}_{d+1}}$  and  $\mathcal{F}_{\text{Bir}_d} \subsetneq \mathcal{F}_{\text{Bir}_{d+1}}$  [18]. Now we state a problem which is related to Problem B-II.

**Problem B-III:** *Is it true that  $\mathcal{F}_{\text{Bir}_d} = \mathcal{F}_{d,0}^*$ ?*

An immediate corollary of Theorem A(1) is:

**Corollary D:** *Let  $d \in \mathbb{N}$ . Then*

$$\mathcal{F}_{Poi_d} \subset \mathcal{F}_{Bir_d} \subset \mathcal{F}_{d,0}^*.$$

Note that  $\mathcal{F}_{Poi_1} \neq \mathcal{F}_{Bir_1}$  [44]. Though we do not know if  $\mathcal{F}_{Bir_d} = \mathcal{F}_{d,0}^*$ , we will show that the two collections coincide “dynamically”, i.e. both of them can be used to characterize higher order almost automorphic points.

**1.2.4. Main results on higher order almost automorphy.** As we said before, Veech studied AA systems, and Veech and Furstenberg gave characterizations of AA systems in [51] and [20] respectively. In this paper we aim to define  $d$ -step AA systems and obtain their characterizations for  $d \in \mathbb{N} \cup \{\infty\}$ . Now we state the main results of the paper. For the definitions when  $d = \infty$  see the following sections.

1.2.4.1.  *$d$ -step AA and proximal relations of order  $d$ .* The following result shows that we can use  $\mathcal{F}_{Poi_d}$ ,  $\mathcal{F}_{Bir_d}$  and  $\mathcal{F}_{d,0}^*$  to characterize regionally proximal pairs of order  $d$ . Precisely, we show in Theorems 7.2.7 and 7.5.1 that: for a minimal t.d.s.  $(X, T)$ ,  $d \in \mathbb{N} \cup \{\infty\}$  and  $x, y \in X$ , the following statements are equivalent: (1)  $(x, y) \in \mathbf{RP}^{[d]}(X, T)$ . (2)  $N(x, U) \in \mathcal{F}_{Poi_d}$  for each neighborhood  $U$  of  $y$ . (3)  $N(x, U) \in \mathcal{F}_{Bir_d}$  for each neighborhood  $U$  of  $y$ . (4)  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ .

1.2.4.2.  *$d$ -step AA and  $SG_d$ -sets.* The notion of  $SG_d$ -sets was introduced by Host and Kra recently [35] to deal with problems related to  $\text{Nil}_d$  Bohr $_0$ -sets. We show that one may use it to characterize regionally proximal pairs of order  $d$ .

Let  $d \geq 1$  be an integer and let  $P = \{p_i\}_i$  be a (finite or infinite) sequence in  $\mathbb{Z}$ . The set of sums with gaps of length less than  $d$  of  $P$  is the set  $SG_d(P)$  of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \dots + \epsilon_n p_n$$

where  $n \geq 1$  is an integer,  $\epsilon_i \in \{0, 1\}$  for  $1 \leq i \leq n$ , the  $\epsilon_i$  are not all equal to 0, and the blocks of consecutive 0's between two 1 have length less than  $d$ . A subset  $A$  of  $\mathbb{Z}$  is an  $SG_d$ -set if  $A = SG_d(P)$  for some infinite sequence in  $\mathbb{Z}$ ; and it is an  $SG_d^*$ -set if  $A \cap SG_d(P) \neq \emptyset$  for every infinite sequence  $P$  in  $\mathbb{Z}$ . Let  $\mathcal{F}_{SG_d}$  be the family generated by all  $SG_d$ -sets. Note that each  $SG_1$ -set is a  $\Delta$ -set, and each  $SG_1^*$ -set is a  $\Delta^*$ -set. The following is the main result of [35]: every  $SG_d^*$ -set is a PW- $\text{Nil}_d$  Bohr $_0$ -set. Host and Kra [35] asked the following question: Is every  $\text{Nil}_d$  Bohr $_0$ -set an  $SG_d^*$ -set?

Though we can not answer this question, we show in Theorems 7.3.2 and 7.5.1 that: for a minimal t.d.s.,  $d \in \mathbb{N} \cup \{\infty\}$  and  $x, y \in X$ ,  $(x, y) \in \mathbf{RP}^{[d]}(X, T)$  if and only if  $N(x, U) \in \mathcal{F}_{SG_d}$  for each neighborhood  $U$  of  $y$ . Combining Theorems 7.2.7 and 7.3.2 we see that  $\text{Nil}_d$  Bohr $_0$ -sets and  $SG_d^*$ -sets are closely related.

1.2.4.3. *Cubic version of multiple Poincaré recurrence sets.* In [34] Host and Kra proved the  $L^2$  convergence of the multiple ergodic average of cubic version. Using it one may define cubic version of multiple Poincaré and Birkhoff recurrence sets. We will show that they can be used to characterize  $\mathbf{RP}^{[d]}$ . For  $d \in \mathbb{N}$ , a subset  $F$  of  $\mathbb{Z}$  is a *Poincaré recurrence set of order  $d$*  if for each measure preserving system  $(X, \mathcal{B}, \mu, T)$  and  $A \in \mathcal{B}$  with positive measure there are  $n_1, \dots, n_d \in \mathbb{Z}$  such that

$FS(\{n_i\}_{i=1}^d) =: \{n_{i_1} + \dots + n_{i_k} : 1 \leq i_1 < \dots < i_k \leq d\} \subset F$  and

$$\mu\left(A \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A\right)\right) > 0.$$

Similarly, we define Birkhoff recurrence sets of order  $d$ . For  $d \in \mathbb{N}$  let  $\mathcal{F}_{P_d}$  (resp.  $\mathcal{F}_{B_d}$ ) be the family of all Poincaré recurrence sets of order  $d$  (resp. the family of all Birkhoff recurrence sets of order  $d$ ). We have the following result proved in Theorems 7.4.5 and 7.4.11 (see also Theorem 7.5.1): for a minimal t.d.s.  $(X, T)$ ,  $d \in \mathbb{N} \cup \{\infty\}$  and  $x, y \in X$ ,  $(x, y) \in \mathbf{RP}^{[d]}(X, T)$  if and only if  $N(x, U) \in \mathcal{F}_{P_d}$  for each neighborhood  $U$  of  $y$  if and only if  $N(x, U) \in \mathcal{F}_{B_d}$  for each neighborhood  $U$  of  $y$ .

1.2.4.4. *Summary of the main results on higher order almost automorphy.* To sum up we have (see Theorem 7.5.1):

**Theorem E:** *Let  $(X, T)$  be a minimal t.d.s. and  $x, y \in X$ . Then the following statements are equivalent for  $d \in \mathbb{N} \cup \{\infty\}$ :*

- (1)  $(x, y) \in \mathbf{RP}^{[d]}$ .
- (2)  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ .
- (3)  $N(x, U) \in \mathcal{F}_{Poi_d}$  for each neighborhood  $U$  of  $y$ .
- (4)  $N(x, U) \in \mathcal{F}_{Bir_d}$  for each neighborhood  $U$  of  $y$ .
- (5)  $N(x, U) \in \mathcal{F}_{SG_d}$  for each neighborhood  $U$  of  $y$ .
- (6)  $N(x, U) \in \mathcal{F}_{B_d}$  for each neighborhood  $U$  of  $y$ .
- (7)  $N(x, U) \in \mathcal{F}_{P_d}$  for each neighborhood  $U$  of  $y$ .

Using the Ramsey property of the families, we can show that one may use  $\mathcal{F}_{Poi_d}^*$ ,  $\mathcal{F}_{Bir_d}^*$  and  $\mathcal{F}_{d,0}$  to characterize  $d$ -step AA. That is, we show in Theorem 8.2.1 that:

**Theorem F:** *Let  $(X, T)$  be a minimal t.d.s.,  $x \in X$  and  $d \in \mathbb{N} \cup \{\infty\}$ . Then the following statements are equivalent:*

- (1)  $x$  is a  $d$ -step AA point.
- (2)  $N(x, V) \in \mathcal{F}_{d,0}$  for each neighborhood  $V$  of  $x$ .
- (3)  $N(x, V) \in \mathcal{F}_{Poi_d}^*$  for each neighborhood  $V$  of  $x$ .
- (4)  $N(x, V) \in \mathcal{F}_{Bir_d}^*$  for each neighborhood  $V$  of  $x$ .

### 1.3. Further questions

It is believed that a “big” subset of integers should contain “good” linear structures, for example arbitrarily long arithmetic progressions. Szemerédi’s Theorem [50] asserts that every positive density subset has this property. In the spirit of Theorem A it is natural to consider the structure of the set of all common differences for a “big” set. To be precise let  $d \geq 1$  and assume that  $S$  is a “big” subset of integers. Then all common differences  $n$  of arithmetic progressions

$$a, a + n, a + 2n, \dots, a + dn$$

with length  $d + 1$  appearing in  $S$  form a set

$$C_d(S) := \{n \in \mathbb{N} : S \cap (S - n) \cap (S - 2n) \cap \dots \cap (S - dn) \neq \emptyset\}.$$

What can we say about the structure of the set  $C_d(S)$ ?

By Green-Tao's result [28], the primes contain arbitrarily long arithmetic progressions. So we ask the question:

**Question 1:** *Is  $C_d(\mathbb{P})$  a  $\text{Nil}_d$  Bohr<sub>0</sub>-set for prime numbers  $\mathbb{P}$ ?*

A direct corollary of Theorems 7.3.2 and 7.5.1 is the following. Assume  $(X, T)$  is minimal,  $x \in X$  and  $d \in \mathbb{N}$ . If  $x$  is  $\mathcal{F}_{SG_d}^*$ -recurrent, or  $\mathcal{F}_{P_d}^*$ -recurrent, or  $\mathcal{F}_{B_d}^*$ -recurrent then it is  $d$ -step AA. Thus, we have the following question.

**Question 2:** Let  $(X, T)$  be a minimal t.d.s.,  $x \in X$ , and  $d \in \mathbb{N}$ . Is it true that  $x$  is  $d$ -step AA if and only if it is  $\mathcal{F}_{SG_d}^*$ -recurrent if and only if it is  $\mathcal{F}_{P_d}^*$ -recurrent if and only if it is  $\mathcal{F}_{B_d}^*$ -recurrent?

Since  $\mathcal{F}_{SG_d}$  does not have the Ramsey property (Appendix A.1), and we do not know if  $\mathcal{F}_{P_d}$  and  $\mathcal{F}_{B_d}$  have the Ramsey property, we can not apply the methods in the proof of Theorem 8.2.1 to solve Question 2. Note that if the question by Host-Kra in [35] has a positive answer, then by using Theorem 8.2.1 Question 2 has a positive answer for  $\mathcal{F}_{SG_d}$ . We note that there are two possible ways to get positive answer of Question 2 for  $\mathcal{F}_{P_d}$  and  $\mathcal{F}_{B_d}$ : (1) prove  $\mathcal{F}_{P_d}$  and  $\mathcal{F}_{B_d}$  have the Ramsey property, (2) prove  $\mathcal{F}_{P_d} \subset \mathcal{F}_{B_d} \subset \mathcal{F}_{d,0}^*$ . Unfortunately, at this moment we can not prove neither of them.

Recall that Veech [51] showed that for a t.d.s.  $(X, T)$ , a point  $x \in X$  is AA if from any sequence  $\{n'_i\} \subset \mathbb{Z}$  one may extract a subsequence  $\{n_i\}$  such that  $\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} T^{n_i - n_j} x = x$ . So we have

**Question 3:** Is there a similar characterization for a  $d$ -step AA point?

Let  $C(X, Y)$  be the collection of all continuous maps from a topological space  $X$  to a topological space  $Y$ . Let  $V$  be a finite dimensional vector space over the complex field  $\mathbb{C}$ . A function  $f \in C(\mathbb{R}, V)$  is said to be *admissible* if it is bounded and uniformly continuous on  $\mathbb{R}$ . Let  $H(f)$  denote the *hull* of  $f$ , i.e. the closure of  $\{f_\tau | \tau \in \mathbb{R}\}$  in the compact open topology, where  $f_\tau(t) = f(t + \tau)$  ( $\tau \in \mathbb{R}$ ). Then by Ascoli's theorem,  $H(f)$  is compact and the time translation  $\Pi_t g = g_t$  ( $g \in H(f)$ ) induces a compact flow  $(H(f), \mathbb{R})$ . For  $d \in \mathbb{N} \cup \{\infty\}$ , we say  $f \in C(\mathbb{R}, \mathbb{C})$  is a  $d$ -step AA function if  $(H(f), \mathbb{R})$  is an almost one-to-one extension of a minimal  $d$ -step nilflow (see [55] for the definition) and  $f \in H(f)$  is a  $d$ -step AA point.

**Question 4:** Is there a differential equation which has a 2-step AA solution and does not have an AA one?

Recall that Host and Kra [35] asked: Is every  $\text{Nil}_d$  Bohr<sub>0</sub>-set an  $SG_d^*$ -set? Using Theorem B, the question of Host and Kra can be reformulated in the following way:

**Question 5:** *Let  $d \in \mathbb{N}$  and  $S$  be an  $SG_d$ -set. Is it true that for any  $k \in \mathbb{N}$ , any  $P_1, \dots, P_k \in SGP_d$  and any  $\epsilon_i > 0$ , there is  $n \in S$  such that*

$$P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)$$

*for all  $i = 1, \dots, k$ ?*

We remark that since a system of order  $d$  is distal, the above question has an affirmative answer for any IP-set.

#### 1.4. Organization of the paper

We organize the paper as follows: In Chapter 2, we give some basic definitions, and particularly we show the equivalence of the Problems B-I, B-II and B-III. In Chapter 3 we recall some basic facts related to nilpotent Lie groups and nilmanifolds, and study the properties of the metric on nilpotent matrix Lie groups. In Chapter 4, we introduce the notions related to generalized polynomials and special generalized polynomials, and give some basic properties. In the next two chapters we prove the main results.

In Chapter 7, we study  $\text{Nil}_d\text{-Bohr}_0$  sets and higher order recurrence sets, and use them to characterize  $\mathbf{RP}^{[d]}$ . In the final chapter, we introduce the notion of  $d$ -step almost automorphy and obtain various characterizations. In the Appendix, we show  $\mathcal{F}_{SG_2}$  does not have the Ramsey property, Theorem 7.1.3 holds for general compact Hausdorff systems and the cubic version of the multiple Poincaré and Birkhoff recurrence sets can be interpreted using intersectiveness.

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## CHAPTER 2

### Preliminaries

In this chapter we introduce some basic notions related to dynamical systems, explain how Bergelson-Host-Kra's result is related to Problem B-II and show the equivalence of the Problems B-I, B-II and B-III.

#### 2.1. Basic notions

**2.1.1. Measurable and topological dynamics.** In this subsection we give some basic notions in ergodic theory and topological dynamics.

2.1.1.1. *Measurable systems.* In this paper, a *measure preserving system* is a quadruple  $(X, \mathcal{X}, \mu, T)$ , where  $(X, \mathcal{X}, \mu)$  is a Lebesgue probability space and  $T : X \rightarrow X$  is an invertible measure preserving transformation.

We write  $\mathcal{I} = \mathcal{I}(T)$  for the  $\sigma$ -algebra  $\{A \in \mathcal{X} : T^{-1}A = A\}$  of invariant sets. A system is *ergodic* if every  $T$ -invariant set has measure either 0 or 1.

2.1.1.2. *Topological dynamical systems.* A *transformation* of a compact metric space  $X$  is a homeomorphism of  $X$  to itself. A *topological dynamical system*, referred to more succinctly as just a t.d.s. or a *system*, is a pair  $(X, T)$ , where  $X$  is a compact metric space and  $T : X \rightarrow X$  is a transformation. We use  $\rho(\cdot, \cdot)$  to denote the metric on  $X$ .

A t.d.s.  $(X, T)$  is *transitive* if there exists some point  $x \in X$  whose orbit  $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$  is dense in  $X$ . The system is *minimal* if the orbit of any point is dense in  $X$ . This property is equivalent to saying that  $X$  and the empty set are the only closed invariant sets in  $X$ .

A *factor* of a t.d.s.  $(X, T)$  is another t.d.s.  $(Y, S)$  such that there exists a continuous and onto map  $\phi : X \rightarrow Y$  satisfying  $S \circ \phi = \phi \circ T$ . In this case,  $(X, T)$  is called an *extension* of  $(Y, S)$  and the map  $\phi$  is called a *factor map*.

2.1.1.3. We also make use of a more general definition of a measurable or topological system. That is, instead of just a single transformation  $T$ , we consider commuting homeomorphisms  $T_1, \dots, T_k$  of  $X$  or a countable abelian group of transformations. We summarize some basic definitions and properties of systems in the classical setting of one transformation. Extensions to the general case are straightforward.

**2.1.2. Families and filters.** Since many statements of the paper are better stated using the notion of a family, we now give the definition. See [1, 21] for more details.

2.1.2.1. *Furstenberg families.* Recall that a collection  $\mathcal{F}$  of subsets of  $\mathbb{Z}$  is a *family* if it is hereditary upward, i.e.  $F_1 \subset F_2$  and  $F_1 \in \mathcal{F}$  imply  $F_2 \in \mathcal{F}$ . A family  $\mathcal{F}$  is called *proper* if it is neither empty nor the entire power set of  $\mathbb{Z}$ , or, equivalently if

$\mathbb{Z} \in \mathcal{F}$  and  $\emptyset \notin \mathcal{F}$ . Any nonempty collection  $\mathcal{A}$  of subsets of  $\mathbb{Z}$  generates a family  $\mathcal{F}(\mathcal{A}) := \{F \subset \mathbb{Z} : F \supset A \text{ for some } A \in \mathcal{A}\}$ .

For a family  $\mathcal{F}$  its *dual* is the family  $\mathcal{F}^* := \{F \subset \mathbb{Z} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}$ . It is not hard to see that  $\mathcal{F}^* = \{F \subset \mathbb{Z} : \mathbb{Z} \setminus F \notin \mathcal{F}\}$ , from which we have that if  $\mathcal{F}$  is a family then  $(\mathcal{F}^*)^* = \mathcal{F}$ .

**2.1.2.2. Filter and Ramsey property.** If a family  $\mathcal{F}$  is closed under finite intersections and is proper, then it is called a *filter*.

A family  $\mathcal{F}$  has the *Ramsey property* if  $A = A_1 \cup A_2 \in \mathcal{F}$  implies that  $A_1 \in \mathcal{F}$  or  $A_2 \in \mathcal{F}$ . It is well known that a proper family has the Ramsey property if and only if its dual  $\mathcal{F}^*$  is a filter [21].

**2.1.2.3. Some important families.** A subset  $S$  of  $\mathbb{Z}$  is *syndetic* if it has a bounded gap, i.e. there is  $N \in \mathbb{N}$  such that  $\{i, i+1, \dots, i+N\} \cap S \neq \emptyset$  for every  $i \in \mathbb{Z}$ . The collection of all syndetic subsets is denoted by  $\mathcal{F}_s$ .

The *upper Banach density* and *lower Banach density* of  $S$  are

$$BD^*(S) = \limsup_{|I| \rightarrow \infty} \frac{|S \cap I|}{|I|}, \text{ and } BD_*(S) = \liminf_{|I| \rightarrow \infty} \frac{|S \cap I|}{|I|},$$

where  $I$  ranges over intervals of  $\mathbb{Z}$ , while the *upper density* of  $S$  and the *lower density* of  $S$  are

$$D^*(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap [-n, n]|}{2n+1}, \text{ and } D_*(S) = \liminf_{n \rightarrow \infty} \frac{|S \cap [-n, n]|}{2n+1}.$$

If  $D^*(S) = D_*(S)$ , then we say the *density* of  $S$  is  $D(S) = D^*(S) = D_*(S)$ . Let  $\mathcal{F}_{pubd} = \{S \subset \mathbb{Z}_+ : BD^*(S) > 0\}$  and  $\mathcal{F}_{pd} = \{S \subset \mathbb{Z}_+ : D^*(S) > 0\}$ .

Let  $\{b_i\}_{i \in I}$  be a finite or infinite sequence in  $\mathbb{Z}$ . One defines

$$FS(\{b_i\}_{i \in I}) = \left\{ \sum_{i \in \alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \right\}.$$

$F$  is an *IP set* if it contains some  $FS(\{p_i\}_{i=1}^\infty)$ , where  $p_i \in \mathbb{Z}$ . The collection of all IP sets is denoted by  $\mathcal{F}_{ip}$ . A subset of  $\mathbb{Z}$  is an *IP\*-set* if it intersects any IP-set. It is known that the family of all IP\*-sets is a filter and each IP\*-set is syndetic [21].

If  $I$  is finite, then one says  $FS(\{p_i\}_{i \in I})$  is a *finite IP set*. The collection of all sets containing finite IP sets with arbitrarily long lengths is denoted by  $\mathcal{F}_{fip}$ .

## 2.2. Bergelson-Host-Kra' Theorem and the proof of Theorem A(2)

In this section we explain how Bergelson-Host-Kra's result in [5] is related to Problem B-II. First we need some definitions.

**DEFINITION 2.2.1.** Let  $d \geq 1$  be an integer and let  $X = G/\Gamma$  be a  $d$ -step nilmanifold. Let  $\phi$  be a continuous real (or complex) valued function on  $X$  and let  $a \in G$  and  $b \in X$ . The sequence  $\{\phi(a^n \cdot b)\}$  is called a *basic  $d$ -step nilsequence*. A  *$d$ -step nilsequence* is a uniform limit of basic  $d$ -step nilsequences.

For the definition of nilmanifolds see Chapter 3.

DEFINITION 2.2.2. Let  $\{a_n : n \in \mathbb{Z}\}$  be a bounded sequence. We say that  $a_n$  tends to zero in uniform density, and we write  $\text{UD-Lim } a_n = 0$ , if

$$\lim_{N \rightarrow +\infty} \sup_{M \in \mathbb{Z}} \frac{1}{N} \sum_{n=M}^{M+N-1} |a_n| = 0.$$

Equivalently,  $\text{UD-Lim } a_n = 0$  if and only if for any  $\epsilon > 0$ , the set  $\{n \in \mathbb{Z} : |a_n| > \epsilon\}$  has upper Banach density zero. Now we state their result.

THEOREM 2.2.3 (Bergelson-Host-Kra). [5, Theorem 1.9] Let  $(X, \mathcal{X}, \mu, T)$  be an ergodic system, let  $f \in L^\infty(\mu)$  and let  $d \geq 1$  be an integer. The sequence  $\{I_f(d, n)\}$  is the sum of a sequence tending to zero in uniform density and a  $d$ -step nilsequence, where

$$(2.1) \quad I_f(d, n) = \int f(x) f(T^n x) \dots f(T^{dn} x) d\mu(x).$$

Note that in Theorem 2.2.3 the decomposition of  $\{I_f(d, n)\}$  is unique, and if  $f$  is a real-valued function then the corresponding nilsequence is also a real-valued sequence. By Theorem 2.2.3, for any  $A \in \mathcal{X}$

$$(2.2) \quad \{I_{1_A}(d, n)\} = \{\mu(A \cap T^{-n} A \cap \dots \cap T^{-dn} A)\} = F_d + N,$$

where  $F_d$  is a  $d$ -step nilsequence and  $N$  tending to zero in uniform density. Regard  $F_d$  as a function  $F_d : \mathbb{Z} \rightarrow \mathbb{R}$ . By [38] there is a system  $(Z, S)$  of order  $d$ ,  $x_0 \in Z$  and a continuous function  $\phi \in C(Z)$  such that

$$F_d(n) = \phi(S^n x_0).$$

We claim that  $\phi(x_0) > 0$  if  $\mu(A) > 0$ . Assume that contrary that  $\phi(x_0) \leq 0$ . By [22] or [7, Theorem 6.15] there is  $c > 0$  such that

$$\{n \in \mathbb{Z} : \mu(A \cap T^{-n} A \cap \dots \cap T^{-dn} A) > c\}$$

is an  $IP^*$ -set. On the other hand there is a small neighborhood  $V$  of  $x_0$  such that  $\phi(x) < \frac{1}{2}c$  for each  $x \in V$  by the continuity of  $\phi$ . It is known that  $N(x_0, V)$  is an  $IP^*$ -set ([21]) since  $(Z, S)$  is distal ([3, Ch 4, Theorem 3] or [45]). This contradicts (2.2) by the facts that the family of  $IP^*$ -sets is a filter, each  $IP^*$ -set is syndetic and  $N(n)$  tends to zero in uniform density. That is, we have shown that  $\phi(x_0) > 0$  if  $\mu(A) > 0$ .

For each  $\epsilon > 0$ ,  $\{n \in \mathbb{Z} : \phi(S^n x_0) > \phi(x_0) - \frac{1}{2}\epsilon\}$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set. Since  $\{n \in \mathbb{Z} : |N(n)| > \frac{1}{2}\epsilon\}$  has zero upper Banach density we have the following corollary

THEOREM 2.2.4. Let  $(X, \mathcal{X}, \mu, T)$  be an ergodic system and  $d \in \mathbb{N}$ . Then for all  $A \in \mathcal{X}$  with  $\mu(A) > 0$  and  $\epsilon > 0$ , the set

$$I = \{n \in \mathbb{Z} : \mu(A \cap T^{-n} A \cap \dots \cap T^{-dn} A) > \phi(x_0) - \epsilon\}$$

is an almost  $\text{Nil}_d$  Bohr<sub>0</sub>-set, i.e. there is some subset  $M$  of  $\mathbb{Z}$  with  $BD^*(M) = 0$  such that  $I \Delta M$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set.

**Proof of Theorem A(2):** It follows by Theorem 2.2.4 that Theorem A(2) holds, since for a minimal system  $(X, T)$ , each invariant measure of  $(X, T)$  is fully supported.

### 2.3. Equivalence of Problems B-I,II,III

In this subsection we explain why Problems B-I,II,III are equivalent. We need Furstenberg correspondence principle.

**2.3.1. Furstenberg correspondence principle.** Let  $\mathcal{F}(\mathbb{Z})$  denote the collection of finite non-empty subsets of  $\mathbb{Z}$ . The following is the well known Furstenberg correspondence principle [21].

**THEOREM 2.3.1** (Topological case). (1) Let  $E \subset \mathbb{Z}$  be a syndetic set. Then there exist a minimal system  $(X, T)$  and a non-empty open subset  $U$  of  $X$  such that

$$\{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} T^{-n}U \neq \emptyset\} \subset \{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} (E - n) \neq \emptyset\}.$$

(2) For any minimal system  $(X, T)$  and any open non-empty subset  $U$  of  $X$ , there is a syndetic set  $E$  of  $\mathbb{Z}$  such that

$$\{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} (E - n) \neq \emptyset\} \subset \{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} T^{-n}U \neq \emptyset\}.$$

**THEOREM 2.3.2** (Measurable case). (1) Let  $E \subset \mathbb{Z}$  with  $BD^*(E) > 0$ . Then there exists a measurable system  $(X, \mathcal{X}, \mu, T)$  and  $A \in \mathcal{X}$  with  $\mu(A) = BD^*(E)$  such that for all  $\alpha \in \mathcal{F}(\mathbb{Z})$

$$BD^*\left(\bigcap_{n \in \alpha} (E - n)\right) \geq \mu\left(\bigcap_{n \in \alpha} T^{-n}A\right).$$

(2) Let  $(X, \mathcal{X}, \mu, T)$  be a measurable system and  $A \in \mathcal{X}$  with  $\mu(A) > 0$ . There is a subset  $E$  of  $\mathbb{Z}$  with  $D^*(E) \geq \mu(A)$  such that

$$\{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} (E - n) \neq \emptyset\} \subset \{\alpha \in \mathcal{F}(\mathbb{Z}) : \mu\left(\bigcap_{n \in \alpha} T^{-n}A\right) > 0\}.$$

**2.3.2. Equivalence of Problems B-I,II,III.** Let  $\mathcal{F}$  be the family generated by all sets of forms  $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$ , with  $(X, T)$  a minimal system,  $U$  a non-empty open subset of  $X$ . Then it is clear from the definition that

$$\mathcal{F}_{Bir_d} = \mathcal{F}^*.$$

**PROPOSITION 2.3.3.** For any  $d \in \mathbb{N}$  the following statements are equivalent.

- (1) For any  $\text{Nil}_d$  Bohr<sub>0</sub>-set  $A$ , there is a syndetic subset  $S$  of  $\mathbb{Z}$  with  $A \supset \{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$ .
- (2) For any  $\text{Nil}_d$  Bohr<sub>0</sub>-set  $A$ , there are a minimal system  $(X, T)$  and a non-empty open subset  $U$  of  $X$  with  $A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$ .
- (3)  $\mathcal{F}_{Bir_d} \subset \mathcal{F}_{d,0}^*$ .

**PROOF.** Let  $d \in \mathbb{N}$  be fixed. (1) $\Rightarrow$ (2). Let  $A$  be a  $\text{Nil}_d$  Bohr<sub>0</sub>-set, then there is a syndetic subset  $S$  of  $\mathbb{Z}$  with  $A \supset \{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$ . For such  $S$  using Theorem 2.3.1, we get that there exist a minimal system  $(X, T)$  and a non-empty open set  $U \subset X$  such that  $\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\} \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$ . Thus  $A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$ .

(2) $\Rightarrow$ (1) follows similarly by the above argument. (2) $\Rightarrow$ (3) follows by the definition. (3) $\Rightarrow$ (2). Since  $\mathcal{F}_{Bir_d} \subset \mathcal{F}_{d,0}^*$  and  $\mathcal{F}_{Bir_d} = \mathcal{F}^*$ , we have that  $\mathcal{F}^* \subset \mathcal{F}_{d,0}^*$  which implies that  $\mathcal{F} \supset \mathcal{F}_{d,0}$ .  $\square$

PROPOSITION 2.3.4. For any  $d \in \mathbb{N}$  the following statements are equivalent.

- (1) For any syndetic set  $S$ ,  $\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set.
- (2) For any minimal system  $(X, T)$ , and any non-empty open subset  $U$  of  $X$ ,  $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set.
- (3)  $\mathcal{F}_{Bir_d} \supset \mathcal{F}_{d,0}^*$ .

PROOF. Let  $d \in \mathbb{N}$  be fixed. (1) $\Rightarrow$ (2). Let  $(X, T)$  be a minimal system and  $U$  be a non-empty open set of  $X$ . By Theorem 2.3.1, there is a syndetic set  $S$  such that

$$\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\} \subset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

By (1),  $\{n \in \mathbb{Z} : S \cap (S - n) \cap \dots \cap (S - dn) \neq \emptyset\}$  is a  $\text{Nil}_d$  Bohr<sub>0</sub>-set, and so is  $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}$ . Similarly, we have (2) $\Rightarrow$ (1). (2) $\Rightarrow$ (3) follows by the definition. (3) $\Rightarrow$ (2) follows by taking  $*$  on both sides of (3).  $\square$



## CHAPTER 3

### Nilsystems

In this chapter we recall some basic facts concerning nilpotent Lie groups and nilmanifolds. Since in the proofs of our main results we need to use the metric of the nilpotent matrix Lie group, we state some basic properties related to the metric. Notice that we follow Green and Tao [29] to define such a metric.

#### 3.1. Nilmanifolds and nilsystems

##### 3.1.1. Nilmanifolds and nilsystems.

3.1.1.1. *Nilpotent groups.* Let  $G$  be a group. For  $g, h \in G$ , we write  $[g, h] = ghg^{-1}h^{-1}$  for the commutator of  $g$  and  $h$  and we write  $[A, B]$  for the subgroup spanned by  $\{[a, b] : a \in A, b \in B\}$ . The commutator subgroups  $G_j$ ,  $j \geq 1$ , are defined inductively by setting  $G_1 = G$  and  $G_{j+1} = [G_j, G]$ . Let  $d \geq 1$  be an integer. We say that  $G$  is *d-step nilpotent* if  $G_{d+1}$  is the trivial subgroup.

3.1.1.2. *Nilmanifolds.* Let  $G$  be a  $d$ -step nilpotent Lie group and  $\Gamma$  a discrete cocompact subgroup of  $G$ , i.e. a uniform subgroup of  $G$ . The compact manifold  $X = G/\Gamma$  is called a *d-step nilmanifold*. The group  $G$  acts on  $X$  by left translations and we write this action as  $(g, x) \mapsto gx$ . The Haar measure  $\mu$  of  $X$  is the unique probability measure on  $X$  invariant under this action. Let  $\tau \in G$  and  $T$  be the transformation  $x \mapsto \tau x$  of  $X$ , i.e the nilrotation induced by  $\tau \in G$ . Then  $(X, T, \mu)$  is called a *d-step nilsystem*. See [12, 46] for the details.

3.1.1.3. *Systems of order d.* We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If  $(X_i, T_i)_{i \in \mathbb{N}}$  are systems with  $\text{diam}(X_i) \leq M < \infty$  and  $\phi_i : X_{i+1} \rightarrow X_i$  are factor maps, the *inverse limit* of the systems is defined to be the compact subset of  $\prod_{i \in \mathbb{N}} X_i$  given by  $\{(x_i)_{i \in \mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$ , which is denoted by  $\varprojlim_{i \in \mathbb{N}} X_i$ . It is a compact metric space endowed with the distance  $\rho(x, y) = \sum_{i \in \mathbb{N}} 1/2^i \rho_i(x_i, y_i)$ . We note that the maps  $\{T_i\}$  induce a transformation  $T$  on the inverse limit.

**DEFINITION 3.1.1.** [Host-Kra-Maass] [36] A system  $(X, T)$  is called a *system of order d*, if it is an inverse limit of  $d$ -step minimal nilsystems.

An  $\infty$ -step nilsystem or a system of order  $\infty$  is an inverse limit of  $d_i$ -step nilsystem, see [13].

Recall that a subset  $A$  of  $\mathbb{Z}$  is a *Nil<sub>d</sub> Bohr<sub>0</sub>-set* if there exist a  $d$ -step nilsystem  $(X, T)$ ,  $x_0 \in X$  and an open neighborhood  $U$  of  $x_0$  such that  $N(x_0, U)$  is contained in  $A$ . As each  $d$ -step nilsystem is distal, so is a system of order  $d$ . Note that each point in a distal system is minimal. Hence by Definition 3.1.1, it is not hard to see

that a subset  $A$  of  $\mathbb{Z}$  is a  $Nil_d$  Bohr<sub>0</sub>-set if and only if there exist a  $d$ -step (minimal) nilsystem  $(X, T)$  (or a *system*  $(X, T)$  of order  $d$ ),  $x_0 \in X$  and an open neighborhood  $U$  of  $x_0$  such that  $N(x_0, U)$  is contained in  $A$ . Note that here we need the fact that the orbit closure of any point in a  $d$ -step nilsystem is a  $d$ -step nilmanifold [45, Theorem 2.21].

**3.1.2. Reduction.** Let  $X = G/\Gamma$  be a nilmanifold. Then there exists a connected, simply connected nilpotent Lie group  $\widehat{G}$  and  $\widehat{\Gamma} \subset \widehat{G}$  a co-compact subgroup such that  $X$  with the action of  $G$  is isomorphic to a submanifold  $\widetilde{X}$  of  $\widehat{X} = \widehat{G}/\widehat{\Gamma}$  representing the action of  $G$  in  $\widehat{G}$ . See [45] for more details.

Thus a subset  $A$  of  $\mathbb{Z}$  is a  $Nil_d$  Bohr<sub>0</sub>-set if and only if there exist a  $d$ -step nilsystem  $(G/\Gamma, T)$  with  $G$  is a connected, simply connected nilpotent Lie group and  $\Gamma$  a co-compact subgroup of  $G$ ,  $x_0 \in X$  and an open neighborhood  $U$  of  $x_0$  such that  $N(x_0, U)$  is contained in  $A$ .

### 3.1.3. Nilpotent Lie group and Mal'cev basis.

3.1.3.1. We will make use of the Lie algebra  $\mathfrak{g}$  of a  $d$ -step nilpotent Lie group  $G$  together with the exponential map  $\exp : \mathfrak{g} \rightarrow G$ . When  $G$  is a connected, simply-connected  $d$ -step nilpotent Lie group the exponential map is a diffeomorphism [12, 46]. In particular, we have a logarithm map  $\log : G \rightarrow \mathfrak{g}$ . Let

$$\exp(X * Y) = \exp(X)\exp(Y), \quad X, Y \in \mathfrak{g}.$$

3.1.3.2. *Campbell-Baker-Hausdorff formula.* The following Campbell-Baker-Hausdorff formula (CBH formula) will be used frequently

$$\begin{aligned} X * Y = & \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{p_i+q_i>0, 1 \leq i \leq n} \frac{(\sum_{i=1}^n (p_i + q_i))^{-1}}{p_1!q_1! \dots p_n!q_n!} \\ & \times (\text{ad } X)^{p_1}(\text{ad } Y)^{q_1} \dots (\text{ad } X)^{p_n}(\text{ad } Y)^{q_n-1}Y, \end{aligned}$$

where  $(\text{ad } X)Y = [X, Y]$ . (If  $q_n = 0$ , the term in the sum is  $\dots (\text{ad } X)^{p_n-1}X$ ; of course if  $q_n > 1$ , or if  $q_n = 0$  and  $p_n > 1$ , then the term is zero.) The low order nonzero terms are well known,

$$\begin{aligned} X * Y = & X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] \\ & - \frac{1}{48}[Y, [X, [X, Y]]] - \frac{1}{48}[X, [Y, [X, Y]]] \\ & + (\text{commutators in five or more terms}). \end{aligned}$$

3.1.3.3. We assume  $\mathfrak{g}$  is the Lie algebra of  $G$  over  $\mathbb{R}$ , and  $\exp : \mathfrak{g} \rightarrow G$  is the exponential map. The *descending central series* of  $\mathfrak{g}$  is defined inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g}; \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = \text{span}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)}\}.$$

Since  $\mathfrak{g}$  is a  $d$ -step nilpotent Lie algebra, we have

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \dots \supset \mathfrak{g}^{(d)} \supset \mathfrak{g}^{(d+1)} = \{0\}.$$

We note that

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}, \quad \forall i, j \in \mathbb{N}.$$



In particular, each  $\mathfrak{g}^{(k)}$  is an ideal in  $\mathfrak{g}$ .

#### 3.1.3.4. *Mal'cev Basis.*

**DEFINITION 3.1.2.** (Mal'cev basis) Let  $G/\Gamma$  be an  $m$ -dimensional nilmanifold (i.e.  $G$  is a  $d$ -step nilpotent Lie group and  $\Gamma$  is a uniform subgroup of  $G$ ) and let  $G = G_1 \supset \dots \supset G_d \supset G_{d+1} = \{e\}$  be the lower central series filtration. A basis  $\mathcal{X} = \{X_1, \dots, X_m\}$  for the Lie algebra  $\mathfrak{g}$  over  $\mathbb{R}$  is called a *Mal'cev basis* for  $G/\Gamma$  if the following four conditions are satisfied:

- (1) For each  $j = 0, \dots, m-1$  the subspace  $\eta_j := \text{Span}(X_{j+1}, \dots, X_m)$  is a Lie algebra ideal in  $\mathfrak{g}$ , and hence  $H_j := \exp \eta_j$  is a normal Lie subgroup of  $G$ .
- (2) For every  $0 < i \leq d$  we have  $G_i = H_{l_{i-1}}$ . Thus  $0 = l_0 < l_1 < \dots < l_{d-1} \leq m-1$ .
- (3) Each  $g \in G$  can be written uniquely as  $\exp(t_1 X_1) \exp(t_2 X_2) \dots \exp(t_m X_m)$ , for  $t_i \in \mathbb{R}$ .
- (4)  $\Gamma$  consists precisely of those elements which, when written in the above form, have all  $t_i \in \mathbb{Z}$ .

Note that such a basis exists when  $G$  is a connected, simply connected  $d$ -step nilpotent Lie group [12, 29, 46].

**3.1.4. Base points.** The following proposition should be well known.

**PROPOSITION 3.1.3.** Let  $X = G/\Gamma$  be a nilmanifold, and  $T$  be a nilrotation induced by  $a \in G$ . Let  $x \in G$  and  $U$  be an open neighborhood of  $x\Gamma$  in  $X$ . Then there are a uniform subgroup  $\Gamma_x \subset G$  and an open neighborhood  $V \subset G/\Gamma_x$  of  $e\Gamma_x$  such that

$$N_T(x\Gamma, U) = N_{T'}(e\Gamma_x, V),$$

where  $T'$  is a nilrotation induced by  $a \in G$  in  $X' = G/\Gamma_x$ .

**PROOF.** Let  $\Gamma_x = x\Gamma x^{-1}$ . Then  $\Gamma_x$  is also a uniform subgroup of  $G$ .

Put  $V = Ux^{-1}$ , where we view  $U$  as the collections of equivalence classes. It is easy to see that  $V \subset G/\Gamma_x$  is open, which contains  $e\Gamma_x$ . Let  $n \in N_T(x\Gamma, U)$  then  $a^n x\Gamma \in U$  which implies that  $a^n x\Gamma x^{-1} \in Ux^{-1} = V$ , i.e.  $n \in N_{T'}(e\Gamma_x, V)$ . The other direction follows similarly.  $\square$

## 3.2. Nilpotent Matrix Lie Group

**3.2.1.** Let  $M_{d+1}(\mathbb{R})$  denote the space of all  $(d+1) \times (d+1)$ -matrices with real entries. For  $A = (A_{ij})_{1 \leq i, j \leq d+1} \in M_{d+1}(\mathbb{R})$ , we define

$$(3.1) \quad \|A\|_\infty = \max_{1 \leq i, j \leq d+1} |A_{ij}|.$$

Then  $\|\cdot\|_\infty$  is a norm on  $M_{d+1}(\mathbb{R})$  and the norm satisfies the inequalities

$$\|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty$$

for  $A, B \in M_{d+1}(\mathbb{R})$ .

**3.2.2.** Let  $\mathbf{a} = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$ . Then corresponding to  $\mathbf{a}$  we define  $\mathbf{M}(\mathbf{a})$  with

$$\mathbf{M}(\mathbf{a}) = \begin{pmatrix} 1 & a_1^1 & a_1^2 & a_1^3 & \dots & a_1^{d-1} & a_1^d \\ 0 & 1 & a_2^1 & a_2^2 & \dots & a_2^{d-2} & a_2^{d-1} \\ 0 & 0 & 1 & a_3^1 & \dots & a_3^{d-3} & a_3^{d-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{d-1}^1 & a_{d-1}^2 \\ 0 & 0 & 0 & 0 & \dots & 1 & a_d^1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

**3.2.3.** Let  $\mathbb{G}_d$  be the (full) upper triangular group

$$\mathbb{G}_d = \{\mathbf{M}(\mathbf{a}) : a_i^k \in \mathbb{R}, 1 \leq k \leq d, 1 \leq i \leq d-k+1\}.$$

The group  $\mathbb{G}_d$  is a  $d$ -step nilpotent group, and it is clear that for  $A \in \mathbb{G}_d$  there exists a unique  $\mathbf{c} = (c_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  such that  $A = \mathbf{M}(\mathbf{c})$ . Let

$$\Gamma = \{\mathbf{M}(\mathbf{h}) : h_i^k \in \mathbb{Z}, 1 \leq k \leq d, 1 \leq i \leq d-k+1\}.$$

Then  $\Gamma$  is a uniform subgroup of  $\mathbb{G}_d$ .

**3.2.4.** Let  $\mathbf{a} = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  and  $\mathbf{b} = (b_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$ . If  $\mathbf{c} = (c_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  such that  $\mathbf{M}(\mathbf{c}) = \mathbf{M}(\mathbf{a})\mathbf{M}(\mathbf{b})$ , then

$$(3.2) \quad c_i^k = \sum_{j=0}^k a_i^{k-j} b_{i+k-j}^j = a_i^k + \left( \sum_{j=1}^{k-1} a_i^{k-j} b_{i+k-j}^j \right) + b_i^k$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d-k+1$ , where we assume  $a_1^0 = a_2^0 = \dots = a_d^0 = 1$  and  $b_1^0 = b_2^0 = \dots = b_d^0 = 1$ .

## CHAPTER 4

### Generalized polynomials

Generalized polynomials have been studied extensively, see for example the remarkable paper by Bergelson and Leibman [6] and references therein. In this chapter we introduce the notions and study the basic properties of (special) generalized polynomials which will be used in the following chapters. Note that our definition of the generalized polynomials is slightly different from the usual one.

#### 4.1. Definitions

**4.1.1.** For a real number  $a \in \mathbb{R}$ , let  $||a|| = \inf\{|a - n| : n \in \mathbb{Z}\}$  and  $\lceil a \rceil = \min\{m \in \mathbb{Z} : |a - m| = ||a||\}$ .

When studying  $\mathcal{F}_{d,0}$  we find that the generalized polynomials appear naturally. Here is the precise definition. Note that we use  $f(n)$  or  $f$  to denote the generalized polynomials.

#### 4.1.2. Generalized polynomials.

**DEFINITION 4.1.1.** Let  $d \in \mathbb{N}$ . We define the *generalized polynomials* of degree  $\leq d$  (denoted by  $\text{GP}_d$ ) by induction. For  $d = 1$ ,  $\text{GP}_1$  is the smallest collection of functions from  $\mathbb{Z}$  to  $\mathbb{R}$  containing  $\{h_a : a \in \mathbb{R}\}$  with  $h_a(n) = an$  for any  $n \in \mathbb{Z}$ , which is closed under taking  $\lceil \cdot \rceil$ , multiplying by a constant and the finite sums.

Assume that  $\text{GP}_i$  is defined for  $i < d$ . Then  $\text{GP}_d$  is the smallest collection of functions from  $\mathbb{Z}$  to  $\mathbb{R}$  containing  $\text{GP}_i$  with  $i < d$ , functions of the forms

$$a_0 n^{p_0} \lceil f_1(n) \rceil \dots \lceil f_k(n) \rceil$$

(with  $a_0 \in \mathbb{R}$ ,  $p_0 \geq 0$ ,  $k \geq 0$ ,  $f_l \in \text{GP}_{p_l}$  and  $\sum_{l=0}^k p_l = d$ ), which is closed under taking  $\lceil \cdot \rceil$ , multiplying by a constant and the finite sums. Let  $\text{GP} = \cup_{i=1}^{\infty} \text{GP}_i$ .

For example,  $a_1 \lceil a_2 \lceil a_3 n \rceil \rceil + b_1 n \in \text{GP}_1$ , and  $a_1 \lceil a_2 n^2 \rceil + b_1 \lceil b_2 n \lceil b_3 n \rceil \rceil + c_1 n^2 + c_2 n \in \text{GP}_2$ , where  $a_i, b_i, c_i \in \mathbb{R}$ . Note that if  $f \in \text{GP}$  then  $f(0) = 0$ .

**4.1.3. Special generalized polynomials.** Since generalized polynomials are very complicated, we will specify a subclass of them, called the *special generalized polynomials* which will be used in our proofs of the main results. To do this, we need some notions.

For  $a \in \mathbb{R}$ , we define  $L(a) = a$ . For  $a_1, a_2 \in \mathbb{R}$  we define  $L(a_1, a_2) = a_1 \lceil L(a_2) \rceil$ . Inductively, for  $a_1, a_2, \dots, a_\ell \in \mathbb{R}$  ( $\ell \geq 2$ ) we define

$$(4.1) \quad L(a_1, a_2, \dots, a_\ell) = a_1 \lceil L(a_2, a_3, \dots, a_\ell) \rceil.$$

For example,  $L(a_1, a_2, a_3) = a_1 \lceil a_2 \lceil a_3 \rceil \rceil$ .

We give now the precise definition of special generalized polynomials.

**DEFINITION 4.1.2.** For  $d \in \mathbb{N}$  we define *special generalized polynomials of degree  $\leq d$* , denoted by  $\text{SGP}_d$  as follows.  $\text{SGP}_d$  is the collection of generalized polynomials of the forms  $L(n^{j_1}a_1, \dots, n^{j_\ell}a_\ell)$ , where  $1 \leq \ell \leq d$ ,  $a_1, \dots, a_\ell \in \mathbb{R}$ ,  $j_1, \dots, j_\ell \in \mathbb{N}$  with  $\sum_{t=1}^\ell j_t \leq d$ .

Thus  $\text{SGP}_1 = \{an : a \in \mathbb{R}\}$ ,  $\text{SGP}_2 = \{an^2, bn \lceil cn \rceil, en : a, b, c, e \in \mathbb{R}\}$  and  $\text{SGP}_3 = \text{SGP}_2 \cup \{an^3, an \lceil bn^2 \rceil, an^2 \lceil bn \rceil, an \lceil bn \lceil cn \rceil \rceil : a, b, c \in \mathbb{R}\}$ .

**4.1.4.  $\mathcal{F}_{GP_d}$  and  $\mathcal{F}_{SGP_d}$ .** Let  $\mathcal{F}_{GP_d}$  be the family generated by the sets of forms

$$\bigcap_{i=1}^k \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\},$$

where  $k \in \mathbb{N}$ ,  $P_i \in GP_d$ , and  $\epsilon_i > 0$ ,  $1 \leq i \leq k$ . Note that  $P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)$  if and only if  $\|P_i(n)\| < \epsilon_i$ .

Let  $\mathcal{F}_{SGP_d}$  be the family generated by the sets of forms

$$\bigcap_{i=1}^k \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\},$$

where  $k \in \mathbb{N}$ ,  $P_i \in SGP_d$ , and  $\epsilon_i > 0$ ,  $1 \leq i \leq k$ . Note that from the definition both  $\mathcal{F}_{GP_d}$  and  $\mathcal{F}_{SGP_d}$  are filters; and  $\mathcal{F}_{SGP_d} \subset \mathcal{F}_{GP_d}$ .

## 4.2. Basic properties of generalized polynomials

**4.2.1.** The following lemmas lead a way to simplify the generalized polynomials. For  $f \in GP$  we let  $f^* = -\lceil f \rceil$ .

**LEMMA 4.2.1.** Let  $c \in \mathbb{R}$  and  $f_1, \dots, f_k \in GP$  with  $k \in \mathbb{N}$ . Then

$$c \lceil f_1 \rceil \dots \lceil f_k \rceil = c(-1)^k \prod_{i=1}^k (f_i - \lceil f_i \rceil) - c(-1)^k \sum_{\substack{i_1, \dots, i_k \in \{1, *\} \\ (i_1, \dots, i_k) \neq (*, \dots, *)}} f_1^{i_1} \dots f_k^{i_k}.$$

In particular, if  $k = 2$  one has that

$$c \lceil f_1 \rceil \lceil f_2 \rceil = cf_1 \lceil f_2 \rceil - cf_1 f_2 + cf_2 \lceil f_1 \rceil + c(f_1 - \lceil f_1 \rceil)(f_2 - \lceil f_2 \rceil).$$

**PROOF.** Expanding  $\prod_{i=1}^k (f_i - \lceil f_i \rceil)$  we get that

$$\prod_{i=1}^k (f_i - \lceil f_i \rceil) = \sum_{i_1, \dots, i_k \in \{1, *\}} f_1^{i_1} \dots f_k^{i_k}.$$

So we have

$$c \lceil f_1 \rceil \dots \lceil f_k \rceil = c(-1)^k \prod_{i=1}^k (f_i - \lceil f_i \rceil) - c(-1)^k \sum_{\substack{i_1, \dots, i_k \in \{1, *\} \\ (i_1, \dots, i_k) \neq (*, \dots, *)}} f_1^{i_1} \dots f_k^{i_k}.$$

□

Let  $c = 1$  in Lemma 4.2.1 we have

LEMMA 4.2.2. Let  $f_1, f_2, \dots, f_k \in GP$ . Then

$$f_1[f_2] \dots [f_k] = (-1)^{k-1} \prod_{i=1}^k (f_i - [f_i]) + (-1)^k \sum_{\substack{i_1, \dots, i_k \in \{1, *\} \\ (i_1, \dots, i_k) \neq (1, *, \dots, *)}} f_1^{i_1} \dots f_k^{i_k}.$$

In particular, if  $k = 2$  one has that

$$f_1[f_2] = [f_1][f_2] + f_1 f_2 - f_2[f_1] - (f_1 - [f_1])(f_2 - [f_2]).$$

Let  $k = 1$  in Lemma 4.2.1 we have

LEMMA 4.2.3. Let  $c \in \mathbb{R}$  and  $f \in GP$ . Then  $c[f] = cf - c(f - [f])$ .

**4.2.2.** In the next subsection we will show that  $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$ . To do this we use induction. To make the proof clear, first we give some results under the assumption

$$(4.2) \quad \mathcal{F}_{GP_{d-1}} \subset \mathcal{F}_{SGP_{d-1}}.$$

DEFINITION 4.2.4. Let  $r \in \mathbb{N}$  with  $r \geq 2$ . We define

$$\mathcal{SW}_r = \left\{ \prod_{i=1}^{\ell} (w_i(n) - [w_i(n)]) : \ell \geq 2, r_i \geq 1, w_i(n) \in GP_{r_i} \text{ and } \sum_{i=1}^{\ell} r_i \leq r \right\}$$

and

$$\mathcal{W}_r = \mathbb{R}\text{-Span}\{\mathcal{SW}_r\},$$

that is,

$$\mathcal{W}_r = \left\{ \sum_{j=1}^{\ell} a_j p_j(n) : \ell \geq 1, a_j \in \mathbb{R}, p_j(n) \in \mathcal{SW}_r \text{ for each } j = 1, 2, \dots, \ell \right\}.$$

LEMMA 4.2.5. Under the assumption (4.2), for any  $p(n) \in \mathcal{W}_d$  and  $\epsilon > 0$  one has

$$\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{SGP_{d-1}}.$$

PROOF. Since  $\mathcal{F}_{SGP_d}$  is a filter, it is sufficient to show that for any  $p(n) = aq(n)$  and  $\frac{1}{2} > \delta > 0$  with  $q(n) \in \mathcal{SW}_d$  and  $a \in \mathbb{R}$ ,

$$\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\delta, \delta)\} \in \mathcal{F}_{SGP_{d-1}}.$$

Note that as  $q(n) \in \mathcal{SW}_d$ , there exist  $\ell \geq 2, r_i \geq 1, w_i(n) \in GP_{r_i}$  and  $\sum_{i=1}^{\ell} r_i \leq d$  such

that  $q(n) = \prod_{i=1}^{\ell} (w_i(n) - [w_i(n)])$ . Since  $\ell \geq 2$ , one has  $r_1 \leq d - 1$  and so  $w_1(n) \in GP_{d-1}$ . By the assumption (4.2),  $\{n \in \mathbb{Z} : w_1(n) \pmod{\mathbb{Z}} \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\} \in \mathcal{F}_{SGP_{d-1}}$ . By the inequality  $|p(n)| \leq |a||w_1(n) - [w_1(n)]|$  for  $n \in \mathbb{Z}$ , we get that

$$\begin{aligned} \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\delta, \delta)\} &\supset \{n \in \mathbb{Z} : |w_1(n) - [w_1(n)]| \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\} \\ &= \{n \in \mathbb{Z} : w_1(n) \pmod{\mathbb{Z}} \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\}. \end{aligned}$$

Thus  $\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\delta, \delta)\} \in \mathcal{F}_{SGP_{d-1}}$  since  $\{n \in \mathbb{Z} : w_1(n) \pmod{\mathbb{Z}} \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\} \in \mathcal{F}_{SGP_{d-1}}$ .  $\square$

DEFINITION 4.2.6. Let  $r \in \mathbb{N}$  with  $r \geq 2$ . For  $q_1(n), q_2(n) \in GP_r$  we define

$$q_1(n) \simeq_r q_2(n)$$

if there exist  $h_1(n) \in GP_{r-1}$  and  $h_2(n) \in \mathcal{W}_r$  such that

$$q_2(n) = q_1(n) + h_1(n) + h_2(n) \pmod{\mathbb{Z}}$$

for all  $n \in \mathbb{Z}$ .

LEMMA 4.2.7. Let  $p(n) \in GP_r$  and  $q(n) \in GP_t$  with  $r, t \in \mathbb{N}$ .

- (1)  $p(n)[q(n)] \simeq_{r+t} (p(n) - [p(n)])q(n)$ .
- (2) If  $q_1(n), q_2(n), \dots, q_k(n) \in GP_t$  such that  $q(n) = \sum_{i=1}^k q_i(n)$ , then

$$p(n)[q(n)] \simeq_{r+t} \sum_{i=1}^k p(n)[q_i(n)].$$

PROOF. (1) follows from Lemma 4.2.2 and (2) follows from (1).  $\square$

DEFINITION 4.2.8. For  $r \in \mathbb{N}$ , we define

$$GP'_r = \{p \in GP_r : \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{SGP_r} \text{ for any } \epsilon > 0\}.$$

PROPOSITION 4.2.9. Let  $r, k \in \mathbb{N}$ .

- (1) For  $p(n) \in GP_r$ ,  $p(n) \in GP'_r$  if and only if  $-p(n) \in GP'_r$ .
- (2) If  $p_1(n), p_2(n), \dots, p_k(n) \in GP'_r$  then

$$p(n) = p_1(n) + p_2(n) + \dots + p_k(n) \in GP'_r.$$

- (3)  $\mathcal{F}_{GP_d} \subset \mathcal{F}_{SGP_d}$  if and only if  $GP'_d = GP_d$ .

PROOF. (1) can be verified directly. (2) follows from the fact that for each  $\epsilon > 0$ ,  $\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \supset \cap_{i=1}^k \{n \in \mathbb{Z} : p_i(n) \pmod{\mathbb{Z}} \in (-\epsilon/k, \epsilon/k)\}$ . (3) follows from the definition of  $GP'_d$ .  $\square$

LEMMA 4.2.10. Let  $p(n), q(n) \in GP_d$  with  $p(n) \simeq_d q(n)$ . Under the assumption (4.2),  $p(n) \in GP'_d$  if and only if  $q(n) \in GP'_d$ .

PROOF. It follows from Lemma 4.2.5 and the fact that  $\mathcal{F}_{SGP_d}$  is a filter.  $\square$

**4.2.3.**  $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$ .

THEOREM 4.2.11.  $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$  for each  $d \in \mathbb{N}$ .

PROOF. It is easy to see that  $\mathcal{F}_{SGP_d} \subset \mathcal{F}_{GP_d}$ . So it remains to show  $\mathcal{F}_{GP_d} \subset \mathcal{F}_{SGP_d}$ . That is, if  $A \in \mathcal{F}_{GP_d}$  then there is  $A' \in \mathcal{F}_{SGP_d}$  with  $A \supset A'$ . We will use induction to show the theorem.

Assume first  $d = 1$ . In this case we let  $GP_1(0) = \{g_a : a \in \mathbb{R}\}$ , where  $g_a(n) = an$  for each  $n \in \mathbb{Z}$ . Inductively if  $GP_1(0), \dots, GP_1(k)$  have been defined then  $f \in GP_1(k+1)$  if and only if  $f \in GP_1 \setminus (\bigcup_{j=0}^k GP_1(j))$  and there are  $k+1$   $\lceil \quad \rceil$  in  $f$ . It is

clear that  $GP_1 = \cup_{k=1}^{\infty} GP_1(k)$ . If  $f \in GP_1(0)$  then it is clear that  $f \in GP'_1$ . Assume that  $GP_1(0), \dots, GP_1(k) \subset GP'_1$  for some  $k \in \mathbb{Z}_+$ .

Let  $f \in GP_1(k+1)$ . We are going to show that  $f \in GP'_1$ . If  $f = f_1 + f_2$  with  $f_1, f_2 \in \cup_{i=0}^k GP_1(i)$ , then by the above assumption and Proposition 4.2.9 we conclude that  $f \in GP'_1$ . The remaining case is  $f = c[f_1] + f_2$  with  $c \in \mathbb{R} \setminus \{0\}$ ,  $f_1 \in GP_1(k)$ , and  $f_2 \in GP_1(0)$ . By Proposition 4.2.9 and the fact  $GP_1(0) \subset GP'_1$ ,  $f \in GP'_1$  if and only if  $c[f_1] \in GP'_1$ . So it remains to show  $c[f_1] \in GP'_1$ . By Lemma 4.2.3 we have  $c[f_1] = cf_1 - c(f_1 - [f_1])$ . It is clear that  $cf_1 \in GP_1(k) \subset GP'_1$  since  $f_1 \in GP_1(k) \subset GP'_1$ . For any  $\epsilon > 0$  since

$$\{n \in \mathbb{Z} : ||-c(f_1(n) - [f_1(n)])|| < \epsilon\} \supset \left\{n \in \mathbb{Z} : ||f_1(n)|| < \frac{\epsilon}{1+|c|}\right\},$$

it implies that  $-c(f_1 - [f_1]) \in GP'_1$ . By Proposition 4.2.9 again we conclude that  $c[f_1] \in GP'_1$ . Hence  $f \in GP'_1$ . Thus  $GP_1 \subset GP'_1$  and we are done for the case  $d = 1$  by Proposition 4.2.9 (3).

Assume that we have proved  $\mathcal{F}_{GP_{d-1}} \subset \mathcal{F}_{SGP_{d-1}}$   $d \geq 2$ , i.e. the assumption (4.2) holds. We define  $GP_d(k)$  with  $k = 0, 1, 2, \dots$ . First  $f \in GP_d(0)$  if and only if there is no  $[ ]$  in  $f$ , i.e.  $f$  is the usual polynomial of degree  $\leq d$ . Inductively if  $GP_d(0), \dots, GP_d(k)$  have been defined then  $f \in GP_{k+1}$  if and only if  $f \in GP_d \setminus (\cup_{j=0}^k GP_d(j))$  and there are  $k+1$   $[ ]$  in  $f$ . It is clear that  $GP_d = \cup_{k=0}^{\infty} GP_d(k)$ . We now show  $GP_d(k) \subset GP'_d$  by induction on  $k$ .

Let  $f$  be an ordinary polynomial of degree  $\leq d$ . Then  $f(n) = a_0 n^d + f_1(n) \simeq_d a_0 n^d$  with  $f_1 \in GP_{d-1}$ . By Lemma 4.2.10,  $f \in GP'_d$  since  $a_0 n^d \in SGP_d \subset GP'_d$ . This shows  $GP_d(0) \subset GP'_d$ . Now assume that for some  $k \in \mathbb{Z}_+$  we have proved

$$(4.3) \quad \bigcup_{i=0}^k GP_d(i) \subset GP'_d.$$

Let  $f \in GP_d(k+1)$ . We are going to show that  $f \in GP'_d$ . If  $f = f_1 + f_2$  with  $f_1, f_2 \in \cup_{i=0}^k GP_d(i)$ , then by the assumption (4.3) and Proposition 4.2.9 (2) we conclude that  $f \in GP'_d$ . The remaining case is that  $f$  can be expressed as the sum of a function in  $GP_d(0)$  and a function  $g \in GP_d(k+1)$  having the form of

- (1)  $g = c[f_1] \dots [f_l]$  with  $c \neq 0$ ,  $l \geq 1$  or
- (2)  $g = g_1(n)[g_2(n)] \dots [g_l(n)]$  for any  $n \in \mathbb{Z}$  with  $g_1(n) \in SGP_r$  and  $r < d$ .

Since  $GP_d(0) \subset GP'_d$ ,  $f \in GP'_d$  if and only if  $g \in GP'_d$  by Proposition 4.2.9. It remains to show that  $g \in GP'_d$ . There are two cases.

Case (1):  $g = c[f_1] \dots [f_l]$  with  $c \neq 0$ ,  $l \geq 1$ .

If  $l = 1$ , then  $g = c[f_1]$  with  $f_1 \in GP_d(k)$ . By Lemma 4.2.3 we have  $c[f_1] = cf_1 - c(f_1 - [f_1])$ . It is clear that  $cf_1 \in GP_d(k) \subset GP'_d$  since  $f_1 \in GP_d(k) \subset GP'_d$ . For any  $\epsilon > 0$  since

$$\{n \in \mathbb{Z} : ||-c(f_1(n) - [f_1(n)])|| < \epsilon\} \supset \left\{n \in \mathbb{Z} : ||f_1(n)|| < \frac{\epsilon}{1+|c|}\right\},$$

it implies that  $-c(f_1 - [f_1]) \in GP'_d$ . By Proposition 4.2.9 again we conclude that  $g = c[f_1] \in GP'_d$ .

If  $l \geq 2$ , using Lemmas 4.2.1 and 4.2.5 we get that

$$c[f_1] \dots [f_l] \simeq_d -c(-1)^l \sum_{\substack{i_1, \dots, i_l \in \{1, *\} \\ (i_1, \dots, i_l) \neq (*, \dots, *)}} f_1^{i_1} \dots f_l^{i_l}.$$

Since each term of the right side is in  $GP_d(k)$ ,  $g \in GP'_d$  by Lemma 4.2.10, the assumption (4.3) and Proposition 4.2.9 (2).

Case (2):  $g = g_1(n)[g_2(n)] \dots [g_l(n)]$  for any  $n \in \mathbb{Z}$  with  $g_1 \in SGP_r$  and  $1 \leq r < d$ .

In this case using Lemmas 4.2.2 and 4.2.5 we get that

$$g_1[g_2] \dots [g_l] \simeq_d (-1)^l \sum_{\substack{i_1, \dots, i_l \in \{1, *\} \\ (i_1, \dots, i_l) \neq (1, *, \dots, *), (*, *, \dots, *)}} g_1^{i_1} \dots g_l^{i_l}.$$

Assume  $i_1, \dots, i_l \in \{1, *\}$  with  $(i_1, \dots, i_l) \neq (1, *, \dots, *), (*, *, \dots, *)$ . If there are at least two 1 appearing in  $(i_1, i_2, \dots, i_l)$ , then  $(-1)^l g_1^{i_1} \dots g_l^{i_l} \in \bigcup_{i=0}^k GP_d(i)$ . Hence

$$(-1)^l g_1^{i_1} \dots g_l^{i_l} \in GP'_d$$

by the assumption (4.3). The remaining situation is that  $i_1 = *$  and there is exact one 1 appearing in  $(i_2, \dots, i_l)$ . In this case,  $(-1)^l g_1^{i_1} \dots g_l^{i_l} \in GP_d(k+1)$  is the finite sum of the forms  $a_1 n^{t_1} [h_1(n)] \dots [h_{l'}(n)]$  with  $t_1 \geq 1$  and  $h_1(n) = g_1(n)$ ; or the forms  $c[h_l] \dots [h_{l_1}]$  or terms in  $GP'_d$ .

If the term has the form  $a_1 n^{t_1} [h_1(n)] \dots [h_{l'}(n)]$  with  $t_1 \geq 1$  and  $h_1(n) = g_1(n)$ , we let  $g_1^{(1)}(n) = a_1 n^{t_1} [h_1(n)] = a_1 n^{t_1} [g_1(n)] \in SGP_{r_1}$ . It is clear  $d \geq r_1 > r$ . If  $r_1 = d$ , then  $a_1 n^{t_1} [h_1(n)] \dots [h_{l'}(n)] = g_1^{(1)}(n) \in GP'_d$  since  $SGP_d \subset GP'_d$ . If  $r_1 < d$ , then we write

$$a_1 n^{t_1} [h_1(n)] \dots [h_{l'}(n)] = g_1^{(1)}(n) [g_2^{(1)}(n)] \dots [g_{l_1}^{(1)}(n)].$$

By using Case (1) we conclude that

$$g \simeq_d \text{finite sum of the forms } g_1^{(1)}(n) [g_2^{(1)}(n)] \dots [g_{l_1}^{(1)}(n)] \text{ and terms in } GP'_d.$$

Repeating the above process finitely many time (at most  $k+1$ -times) we get that  $g \simeq_d$  finite sum of terms in  $GP'_d$ . Thus  $g \in GP'_d$  by Lemma 4.2.10 and Proposition 4.2.9 (2). The proof is now completed.  $\square$



## CHAPTER 5

### Nil Bohr<sub>0</sub>-sets and generalized polynomials: Proof of Theorem B

In this chapter for a given  $d \in \mathbb{N}$  we investigate the relationship between the family of all Nil <sub>$d$</sub>  Bohr<sub>0</sub>-sets and the family generalized by all generalized polynomials of order  $\leq d$ , i.e we will prove Theorem B.

#### 5.1. Proof of Theorem B(1)

In this section, we will prove Theorem B(1), i.e. we will show that if  $A \in \mathcal{F}_{d,0}$  then there are  $k \in \mathbb{N}$ ,  $P_i \in GP_d$  ( $1 \leq i \leq k$ ) and  $\epsilon_i > 0$  such that

$$A \supset \bigcap_{i=1}^k \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\}.$$

We remark that by Section 3.1.2, it is sufficient to consider the case when the group  $G$  is a connected, simply-connected  $d$ -step nilpotent Lie group.

**5.1.1. Notations.** Let  $X = G/\Gamma$  with  $G$  a connected, simply-connected  $d$ -step nilpotent Lie group,  $\Gamma$  a uniform subgroup. Let  $T : X \rightarrow X$  be the nilrotation induced by  $a \in G$ .

Let  $\mathfrak{g}$  be the Lie algebra of  $G$  over  $\mathbb{R}$ , and let  $\exp : \mathfrak{g} \rightarrow G$  be the exponential map. Consider

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \dots \supset \mathfrak{g}^{(d)} \supset \mathfrak{g}^{(d+1)} = \{0\}.$$

Notice that

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}, \quad \forall i, j \in \mathbb{N}.$$

There is a Mal'cev basis  $\mathcal{X} = \{X_1, \dots, X_m\}$  for  $\mathfrak{g}$  with

- (1) For each  $j = 0, \dots, m-1$  the subspace  $\eta_j := \text{Span}(X_{j+1}, \dots, X_m)$  is a Lie algebra ideal in  $\mathfrak{g}$ , and hence  $H_j := \exp \eta_j$  is a normal Lie subgroup of  $G$ .
- (2) For every  $0 < i \leq d$  we have  $G_i = H_{l_{i-1}}$ , where  $0 = l_0 < l_1 < \dots < l_{d-1} < l_d = m$ .
- (3) Each  $g \in G$  can be written uniquely as  $\exp(t_1 X_1) \exp(t_2 X_2) \dots \exp(t_m X_m)$ , for  $t_i \in \mathbb{R}$ .
- (4)  $\Gamma$  consists precisely of those elements which, when written in the above form, have all  $t_i \in \mathbb{Z}$ ,

where  $G = G_1$ ,  $G_{i+1} = [G_i, G]$  with  $G_{d+1} = \{e\}$ . Notice that  $\text{Span}\{X_{l_{i+1}}, \dots, X_m\} = \mathfrak{g}^{(i+1)}$  for  $i = 0, 1, \dots, d-1$ .

DEFINITION 5.1.1. Let  $\{X_1, \dots, X_m\}$  be a Mal'cev bases for  $G/\Gamma$ . Assume that  $P = P(u_1, \dots, u_m)$  is a polynomial. Define the *weighted degree*  $o(u_i)$  of  $u_i$  to be the largest integer  $k$  such that  $X_i$  is contained in  $\mathfrak{g}^{(k)}$ , i.e.  $o(u_i) = j$  if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $1 \leq j \leq d$ . The weighted degree of a monomial in  $u_i$ 's is the sum of the weighted degree of each term, i.e.  $o(u_1^{k_1} \dots u_m^{k_m}) = \sum_{i=1}^m k_i o(u_i)$ . As usual the weighted degree of  $P$  is the maximum of the weighted degrees of monomials in  $P$ .

**5.1.2. Some lemmas.** We need several lemmas. Note that if

$$\exp(t_1 X_1) \dots \exp(t_m X_m) = \exp(u_1 X_1 + \dots + u_m X_m)$$

it is known that [12, 46] each  $t_i$  is a polynomial of  $u_1, \dots, u_m$  and each  $u_i$  is a polynomial of  $t_1, \dots, t_m$ . For our purpose we need to know the precise degree of the polynomials.

LEMMA 5.1.2. Let  $\{X_1, \dots, X_m\}$  be a Mal'cev bases for  $G/\Gamma$ . Assume that

$$\exp(t_1 X_1) \dots \exp(t_m X_m) = \exp(u_1 X_1 + \dots + u_m X_m).$$

Then we have

- (1) Each  $u_i$  is a polynomial in  $t_j$ 's with no constant term such that the weighted degree of the polynomial is no bigger than that of  $u_i$  and the ordinary degree 1 part of this polynomial is exactly  $t_i$  (i.e.  $u_i = t_i$  for  $1 \leq i \leq l_1$  and if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $2 \leq j \leq d$  then  $u_i = t_i + \sum c_{k_1, \dots, k_m, i} t_1^{k_1} \dots t_m^{k_m}$ , where the sum is over all  $0 \leq k_1, \dots, k_m \leq m$  with  $\sum_{j=1}^m k_j o(t_j) \leq o(u_i)$  and there are at least two  $j$ 's with  $k_j \neq 0$ ).
- (2) Each  $t_i$  is a polynomial in  $u_j$ 's with no constant term such that the weighted degree of the polynomial is no bigger than that of  $t_i$  and the ordinary degree 1 part of this polynomial is exactly  $u_i$ .

PROOF. (1). It is easy to see that if  $m = 1$  then  $d = 1$  and (1) holds. So we may assume that  $m \geq 2$ . For  $s \in \{0, 1, \dots, m\}^{\{1, \dots, m\}}$ , let  $\{i_1 < \dots < i_n\}$  be the collection of  $p$ 's with  $s(p) \neq 0$ . Let

$$X_s = [X_{s(i_1)}, [X_{s(i_2)}, \dots, [X_{s(i_{n-1})}, X_{s(i_n)}]]].$$

For each  $0 \leq p \leq m$  let  $k_p(s)$  be the number of  $p$ 's appearing in  $s(1), \dots, s(m)$  (as usual, the cardinality of the empty set is defined as 0). Using the CBH formula  $m-1$  times and the condition  $\mathfrak{g}^{(d+1)} = \{0\}$  it is easy to see that  $(t_1 X_1) * \dots * (t_m X_m)$  is the sum of  $\sum_{i=1}^m t_i X_i$  and the terms

$$\text{constant} \times t_{q_1} \dots t_{q_n} [X_{q_1}, [X_{q_2}, \dots, [X_{q_{n-1}}, X_{q_n}]]], \quad m \geq n \geq 2,$$

i.e.  $\exp(t_1 X_1) \dots \exp(t_m X_m)$  can be written as

$$\exp\left(\sum_{j=1}^m t_j X_j + \sum c'_s t_1^{k_1(s)} \dots t_m^{k_m(s)} X_s\right),$$

where the sum is over all  $s \in \{0, 1, \dots, m\}^{\{1, \dots, m\}}$  and there are at least two  $j$ 's with  $s(j) \neq 0$ . Note that  $X_s \in \mathfrak{g}^{(\sum_{j=1}^m k_j(s) o(t_j))}$ . Let  $X_s = \sum_{j=1}^m c'_{s,j} X_j$ . Thus,

$c'_{s,1}, \dots, c'_{s,i} = 0$  if  $\sum_{j=1}^m k_j(s) o(t_j) > o(t_i)$ . Hence,  $u_i = t_i$  for  $1 \leq i \leq l_1$  and if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $2 \leq j \leq d$  then the coefficient of  $X_i$  is

$$u_i = t_i + \sum c_{k_1, \dots, k_m, i} t_1^{k_1} \dots t_m^{k_m},$$

where the sum is over all  $0 \leq k_1, \dots, k_m \leq m$  with  $\sum_{j=1}^m k_j o(t_j) \leq o(u_i)$  and there are at least two  $j$ 's with  $k_j \neq 0$ .

Note that when  $k_1 o(t_1) + \dots + k_m o(t_m) \leq o(u_i)$  and there are at least two  $j$ 's with  $k_j \neq 0$ , we have that  $k_i = k_{i+1} = \dots = k_m = 0$  and some other restrictions. For example, when  $l_1 + 1 \leq i \leq l_2$ ,  $t_1^{k_1} \dots t_m^{k_m} = t_{i_1} t_{i_2}$  with  $1 \leq i_1, i_2 \leq l_1$ ; and when  $l_2 + 1 \leq i \leq l_3$ ,  $t_1^{k_1} \dots t_m^{k_m} = t_{i_1} t_{i_2} t_{i_3}$  with  $1 \leq i_1, i_2, i_3 \leq l_1$  or  $t_{i_1} t_{i_2}$  with  $1 \leq i_1 \leq l_1$  and  $l_1 + 1 \leq i_2 \leq l_2$ .

(2) It is easy to see that  $t_i = u_i$  for  $1 \leq i \leq l_1$ . If  $d = 1$  (2) holds, and thus we assume that  $d \geq 2$ . We show (2) by induction. We assume that

$$(5.1) \quad t_p = u_p + \sum d_{k'_1, \dots, k'_m, p} u_1^{k'_1} \dots u_m^{k'_m},$$

where the sum is over all  $0 \leq k'_1, \dots, k'_m \leq m$  with  $\sum_{j=1}^m k'_j o(u_j) \leq o(t_p)$  and there are at least two  $j$ 's with  $k'_j \neq 0$  for all  $p$  with  $l_1 + 1 \leq p \leq i$ .

Since

$$u_{i+1} = t_{i+1} + \sum c_{k_1, \dots, k_m, i+1} t_1^{k_1} \dots t_m^{k_m},$$

we have that

$$t_{i+1} = u_{i+1} - \sum c_{k_1, \dots, k_m, i+1} t_1^{k_1} \dots t_m^{k_m},$$

where the sum is over all  $0 \leq k_1, \dots, k_m \leq m$  with  $\sum_{j=1}^m k_j o(t_j) \leq o(t_{i+1})$  and there are at least two  $j$ 's with  $k_j \neq 0$ .

Since  $o(t_{i+1}) \leq o(t_i) + 1$  and there are at least two  $j$ 's with  $k_j \neq 0$ , we have that if  $k_1 o(u_1) + \dots + k_m o(u_m) \leq o(t_{i+1})$  then  $k_j o(u_j) \leq o(t_i)$  for each  $1 \leq j \leq m$ , which implies that  $k_{i+1}, \dots, k_m = 0$ . By the induction each  $t_p$  ( $1 \leq p \leq i$ ) is a polynomial of  $u_1, \dots, u_m$  of the weighted degree at most  $o(t_p)$  (see Equation (5.1)) thus

$$\sum c_{k_1, \dots, k_m, i+1} t_1^{k_1} \dots t_m^{k_m}$$

is a polynomial of  $u_1, \dots, u_m$  of the weighted degree at most  $\sum_{p=1}^m k_p o(u_p) = \sum_{p=1}^m k_p o(t_p) \leq o(t_{i+1})$ . Rearranging the coefficients we get (2). Note that there are at least two  $j$ 's with  $k_j \neq 0$ .  $\square$

LEMMA 5.1.3. Assume that

$$x = \exp(x_1 X_1 + \dots + x_m X_m) \quad \text{and} \quad y = \exp(y_1 X_1) \dots \exp(y_m X_m).$$

Then

$$xy^{-1} = \exp\left(\sum_{i=1}^{l_1} (x_i - y_i) X_i + \sum_{i=l_1+1}^m ((x_i - y_i) + P_{i,1}(\{y_p\}) + P_{i,2}(\{x_p\}, \{y_p\})) X_i\right),$$

where  $P_{i,1}(\{y_p\}), P_{i,2}(\{x_p\}, \{y_p\})$  are polynomials of the weighted degree at most  $o(y_i)$  for each  $l_1 + 1 \leq i \leq m$ . To be precise, we have

$$(5.2) \quad P_{i,1}(\{y_p\}) = - \sum c_{k'_1, \dots, k'_m, i} y_1^{k'_1} \dots y_m^{k'_m},$$

the sum is over all  $0 \leq k'_1, \dots, k'_m \leq m$  with  $\sum_{j=1}^m k'_j o(y_j) \leq o(y_i)$  and there are at least two  $j$ 's with  $k'_j \neq 0$  for all  $i$  with  $l_1 + 1 \leq i \leq m$ . Moreover,

$$(5.3) \quad P_{i,2}(\{x_p\}, \{y_p\}) = \sum e_{k_1, \dots, k_m}^{k'_1, \dots, k'_m} x_1^{k_1} \dots x_m^{k_m} y_1^{k'_1} \dots y_m^{k'_m},$$

the sum is over all  $0 \leq k_j, k'_j \leq m$  with  $\sum_{j=1}^m (k_j + k'_j) o(y_j) \leq o(y_i)$ , and there are at least one  $j$  with  $k_j \neq 0$ , one  $j$  with  $k'_j \neq 0$  for all  $l_1 + 1 \leq i \leq m$ .

PROOF. By Lemma 5.1.2 we have

$$xy^{-1} = \exp(X)\exp(Y)$$

with  $X = \sum_{i=1}^m x_i X_i$  and

$$Y = - \sum_{i=1}^m y_i X_i + \sum_{i=l_1+1}^m P_{i,1}(\{y_p\}) X_i,$$

where

$$P_{i,1}(\{y_p\}) = - \sum c_{k'_1, \dots, k'_m, i} y_1^{k'_1} \dots y_m^{k'_m},$$

the sum is over all  $0 \leq k'_1, \dots, k'_m \leq m$  with  $\sum_{j=1}^m k'_j o(y_j) \leq o(y_i)$  and there are at least two  $j$ 's with  $k'_j \neq 0$  for all  $i$  with  $l_1 + 1 \leq i \leq m$ .

Using the CBH formula we get that

$$\begin{aligned} xy^{-1} &= \exp(X * Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \dots\right) \\ &= \exp\left(\sum_{i=1}^m (x_i - y_i) X_i + \sum_{i=l_1+1}^m (P_{i,1}(\{y_p\}) + P_{i,2}(\{x_p\}, \{y_p\})) X_i\right) \\ &= \exp\left(\sum_{i=1}^{l_1} (x_i - y_i) X_i + \sum_{i=l_1+1}^m ((x_i - y_i) + P_{i,1}(\{y_p\}) + P_{i,2}(\{x_p\}, \{y_p\})) X_i\right) \end{aligned}$$

where

$$P_{i,2}(\{x_p\}, \{y_p\}) = \sum e_{k_1, \dots, k_m}^{k'_1, \dots, k'_m} x_1^{k_1} \dots x_m^{k_m} y_1^{k'_1} \dots y_m^{k'_m},$$

the sum is over all  $0 \leq k_j, k'_j \leq m$  with  $\sum_{j=1}^m (k_j + k'_j) o(y_j) \leq o(y_i)$ , and there are at least one  $j$  with  $k_j \neq 0$ , one  $j$  with  $k'_j \neq 0$  for all  $l_1 + 1 \leq i \leq m$ .

Note that the reason  $P_{i,2}$  has the above form follows from the fact that  $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}, \mathfrak{g}^{(d+1)} = \{0\}$  and a discussion similar to the one used in Lemma 5.1.2.  $\square$

**5.1.3. Proof of Theorem B(1).** Let  $X = G/\Gamma$  with  $G$  a connected, simply-connected  $d$ -step nilpotent Lie group,  $\Gamma$  a uniform subgroup. Let  $T : X \rightarrow X$  be the nilrotation induced by  $a \in G$ . Assume that  $A \supset N(x\Gamma, U)$  with  $x \in G$ ,  $x\Gamma \in U$  and  $U \subset G/\Gamma$  open. By Proposition 3.1.3 we may assume that  $x$  is the unit element  $e$  of  $G$ , i.e.  $A \supset N(e\Gamma, U)$ .

Assume that  $a = \exp(a_1 X_1 + \dots + a_m X_m)$ , where  $a_1, \dots, a_m \in \mathbb{R}$ . Then

$$a^n = \exp(na_1 X_1 + \dots + na_m X_m)$$

for any  $n \in \mathbb{Z}$ . For  $h = \exp(h_1 X_1) \dots \exp(h_m X_m)$ , where  $h_1, \dots, h_m \in \mathbb{R}$ , write

$$a^n h^{-1} = \exp(p_1 X_1 + \dots + p_m X_m) = \exp(w_1 X_1) \dots \exp(w_m X_m).$$

Then by Lemma 5.1.3 with  $x_j, y_j$  are replaced by  $na_j, h_j$  respectively, we have

(1) if  $1 \leq i \leq l_1$ , then  $p_i = na_i - h_i$ , and

(2) if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $2 \leq j \leq d$  then

$$(5.4) \quad p_i = na_i - h_i + P_{i,1}(\{h_p\}) + P_{i,2}(\{na_p\}, \{h_p\}),$$

where  $P_{i,1}$  is defined in (5.2) and  $P_{i,2}$  is defined in (5.3), which satisfy the properties stated there. It is clear that

$$P_{i,2}(\{na_p\}, \{h_p\}) = \sum e_{k_1, \dots, k_m}^{k'_1, \dots, k'_m} n^{k_1 + \dots + k_m} a_1^{k_1} \dots a_m^{k_m} h_1^{k'_1} \dots h_m^{k'_m}.$$

Changing the exponential coordinates to Mal'sev coordinates (Lemma 5.1.2), we get that

(i) if  $1 \leq i \leq l_1$ , then  $w_i = na_i - h_i$ , and

(ii) if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $2 \leq j \leq d$ , then

$$w_i = p_i + \sum d_{k_1, \dots, k_m, i} p_1^{k_1} \dots p_m^{k_m},$$

where the sum is over all  $0 \leq k_1, \dots, k_m \leq m$  with  $\sum_{j=1}^m k_j o(t_j) \leq o(u_i)$  and there are at least two  $j$ 's with  $k_j \neq 0$ . In this case using (5.4) it is not hard to see that

$$w_i = -h_i + Q_i(n, h_1, \dots, h_m)$$

such that  $Q_i$  is a polynomial, and each term of  $Q_i$  is the form of

$$c(k'_1, \dots, k'_m, k_1, \dots, k_m) n^{k'_1 + \dots + k'_m} h_1^{k_1} \dots h_m^{k_m}$$

with  $\sum_{j=1}^m (k'_j + k_j) o(h_j) \leq o(h_i)$  (see the argument of Lemma 5.1.2(2)). Note that if  $k = k'_1 + \dots + k'_m = 0$  then there are at least two  $j$ 's with  $k_j \neq 0$ , and if  $k_1 = \dots = k_m = 0$  then  $k \geq 1$ . This implies that in case (ii) in fact we have

$$w_i = -h_i + Q_i(n, h_1, \dots, h_{i-1}).$$

For a given  $n \in \mathbb{Z}$ , let  $h_i(n) = \lceil na_i \rceil$  if  $1 \leq i \leq l_1$ . Moreover, when  $h_v$  is defined for  $1 \leq v \leq i-1$  we let

$$h_i(n) = \lceil Q_i(n, h_1(n), \dots, h_{i-1}(n)) \rceil$$

if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $2 \leq j \leq d$ . Again a similar argument as in the proof of Lemma 5.1.2(2) shows that  $h_i(n)$  is well defined and is a generalized polynomial of

degree at most  $o(h_i) \leq d$ . For example, if  $l_1 + 1 \leq i \leq l_2$  then

$$p_i = na_i - h_i + \sum_{1 \leq i_1 < i_2 \leq l_1} c(i_1, i_2, i) h_{i_1} h_{i_2} + \sum_{1 \leq j_1 \leq l_1} c(j_1, i) n h_{j_1}.$$

So

$$\begin{aligned} w_i &= na_i - h_i + \sum_{1 \leq i_1 < i_2 \leq l_1} c(i_1, i_2, i) h_{i_1} h_{i_2} + \sum_{1 \leq j_1 \leq l_1} c(j_1, i) n h_{j_1} + \sum_{1 \leq i_1 < i_2 \leq l_1} d(i_1, i_2, i) p_{i_1} p_{i_2} \\ &= na_i - h_i + \sum_{1 \leq i_1 < i_2 \leq l_1} c(i_1, i_2, i) h_{i_1} h_{i_2} + \sum_{1 \leq j_1 \leq l_1} c(j_1, i) n h_{j_1} \\ &\quad + \sum_{1 \leq i_1 < i_2 \leq l_1} d(i_1, i_2, i) (na_{i_1} - h_{i_1})(na_{i_2} - h_{i_2}). \end{aligned}$$

Thus if we let  $h_i(n) = \lceil na_i \rceil$ ,  $1 \leq i \leq l_1$  then if  $l_1 + 1 \leq i \leq l_2$

$$\begin{aligned} h_i(n) &= \lceil na_i + \sum_{1 \leq i_1 < i_2 \leq l_1} c(i_1, i_2, i) \lceil na_{i_1} \rceil \lceil na_{i_2} \rceil + \sum_{1 \leq j_1 \leq l_1} c(j_1, i) n \lceil na_{j_1} \rceil \\ &\quad + \sum_{1 \leq i_1 < i_2 \leq l_1} d(i_1, i_2, i) (na_{i_1} - \lceil na_{i_1} \rceil)(na_{i_2} - \lceil na_{i_2} \rceil) \rceil. \end{aligned}$$

That is,

$$h_i(n) = \lceil na_i + n^2 a'_i + \sum_{1 \leq i_1 < i_2 \leq l_1} c'(i_1, i_2, i) \lceil na_{i_1} \rceil \lceil na_{i_2} \rceil + \sum_{1 \leq j_1 \leq l_1} c'(j_1, i) n \lceil na_{j_1} \rceil \rceil$$

is a generalized polynomial of degree at most 2 in  $n$ .

Next we let  $w_i(n) = na_i - h_i(n) = na_i - \lceil na_i \rceil$  for  $1 \leq i \leq l_1$  and if  $l_{j-1} + 1 \leq i \leq l_j$ ,  $2 \leq j \leq d$ , let

$$\begin{aligned} w_i(n) &= Q_i(n, h_1(n), \dots, h_{i-1}(n)) - h_i(n) \\ &= Q_i(n, h_1(n), \dots, h_{i-1}(n)) - \lceil Q_i(n, h_1(n), \dots, h_{i-1}(n)) \rceil. \end{aligned}$$

The previous argument shows that  $w_i(n)$  is a generalized polynomial of degree at most  $d$ .

Let  $h(n) = \exp(h_1(n)X_1) \dots \exp(h_m(n)X_m)$ . Then  $h(n) \in \Gamma$  and

$$a^n h(n)^{-1} = \exp(w_1(n)X_1) \dots \exp(w_m(n)X_m).$$

Denote by  $\pi$  the quotient map  $\pi : G \rightarrow X$ . Since  $\pi^{-1}(U)$  is open and contains  $e$ , there is some  $0 < \epsilon < \frac{1}{2}$  such that

$$\pi^{-1}(U) \supset \{\exp(t_1 X_1) \dots \exp(t_m X_m) : |t_1|, \dots, |t_m| < \epsilon\} =: V.$$

Then

$$A \supset N(e\Gamma, U) \supset \{n \in \mathbb{Z} : a^n h(n)^{-1} \in V\}.$$

So if  $n \in \bigcap_{i=1}^m \{n \in \mathbb{Z} : w_i(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\}$  then  $n \in \{n \in \mathbb{Z} : a^n h(n)^{-1} \in V\} \subset N(e\Gamma, U) \subset A$ . That is,

$$A \supset \bigcap_{i=1}^m \{n \in \mathbb{Z} : w_i(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\}.$$

This ends the proof of Theorem B(1).

### 5.2. Proof of Theorem B(2)

In this section, we aim to prove Theorem B(2), i.e.  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ . To do this first we make some preparations, then derive some results under the inductive assumption, and finally give the proof. Note that in the construction the nilpotent matrix Lie group is used.

More precisely, to show  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$  we need only to prove  $\mathcal{F}_{d,0} \supset \mathcal{F}_{SGP_d}$  by Theorem 4.2.11. To do this, for a given  $F \in \mathcal{F}_{SGP_d}$  we need to find a  $d$ -step nilsystem  $(X, T)$ ,  $x_0 \in X$  and a neighborhood  $U$  of  $x_0$  such that  $F \supset N(x_0, U)$ . In the process of doing this, we find that it is convenient to consider a finite sum of specially generalized polynomials  $P(n; \alpha_1, \dots, \alpha_r)$  (defined in (5.8)) instead of considering a single specially generalized polynomial. We can prove that  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$  if and only if  $\{n \in \mathbb{Z} : \|P(n; \alpha_1, \dots, \alpha_d)\| < \epsilon\} \in \mathcal{F}_{d,0}$  for any  $\alpha_1, \dots, \alpha_d \in \mathbb{R}$  and  $\epsilon > 0$  (Theorem 5.2.7). We choose  $(X, T)$  as the closure of the orbit of  $\Gamma$  in  $\mathbb{G}_d/\Gamma$  (the nilrotation is induced by a matrix  $A \in \mathbb{G}_d$ ), and consider the most right-corner entry  $z_1^d(n)$  in  $A^n B_n$  with  $B_n \in \Gamma$ . We finish the proof by showing that  $P(n; \alpha_1, \dots, \alpha_d) \simeq_d z_1^d(n)$  and  $\{n \in \mathbb{Z} : \|z_1^d(n)\| < \epsilon\} \in \mathcal{F}_{d,0}$  for any  $\epsilon > 0$ .

**5.2.1. Some preparations.** For a matrix  $A$  in  $\mathbb{G}_d$  we now give a precise formula of  $A^n$ .

LEMMA 5.2.1. Let  $\mathbf{x} = (x_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$ . For  $n \in \mathbb{N}$ , assume that  $\mathbf{x}(n) = (x_i^k(n))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  satisfies  $\mathbf{M}(\mathbf{x}(n)) = \mathbf{M}(\mathbf{x})^n$ , then

$$(5.5) \quad x_i^k(n) = \binom{n}{1} P_1(\mathbf{x}; i, k) + \binom{n}{2} P_2(\mathbf{x}; i, k) + \dots + \binom{n}{k} P_k(\mathbf{x}; i, k)$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ , where  $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$  for  $n, k \in \mathbb{N}$  and

$$P_\ell(\mathbf{x}; i, k) = \sum_{\substack{(s_1, s_2, \dots, s_\ell) \in \{1, 2, \dots, k\}^\ell \\ s_1 + s_2 + \dots + s_\ell = k}} x_i^{s_1} x_{i+s_1}^{s_2} x_{i+s_1+s_2}^{s_3} \dots x_{i+s_1+s_2+\dots+s_{\ell-1}}^{s_\ell}$$

for  $1 \leq k \leq d$ ,  $1 \leq i \leq d - k + 1$  and  $1 \leq \ell \leq k$ .

PROOF. Let  $x_i^0 = 1$  and  $x_i^0(m) = 1$  for  $1 \leq i \leq d$  and  $m \in \mathbb{N}$ . By (3.2), it is not hard to see that

$$(5.6) \quad x_i^k(m+1) = \sum_{j=0}^k x_i^{k-j}(m) \cdot x_{i+k-j}^j$$

for  $1 \leq k \leq d$ ,  $1 \leq i \leq d - k + 1$  and  $m \in \mathbb{N}$ .

Now we do induction for  $k$ . When  $k = 1$ ,  $x_i^1(1) = x_i^1$  and  $x_i^1(m+1) = x_i^1(m) + x_i^1$  for  $m \in \mathbb{N}$  by (5.6). Hence  $x_i^1(n) = nx_i^1 = \binom{n}{1} P_1(\mathbf{x}; i, 1)$ . That is, (5.5) holds for each  $1 \leq i \leq d$  and  $n \in \mathbb{N}$  if  $k = 1$ .

Assume that  $1 \leq \ell \leq d - 1$ , and (5.5) holds for each  $1 \leq k \leq \ell$ ,  $1 \leq i \leq d - k + 1$  and  $n \in \mathbb{N}$ . For  $k = \ell + 1$ , we make induction on  $n$ . When  $n = 1$  it is clear

$$x_i^k(1) = x_i^k = \binom{1}{1} P_1(\mathbf{x}; i, k) + \binom{1}{2} P_2(\mathbf{x}; i, k) + \dots + \binom{1}{k} P_k(\mathbf{x}; i, k)$$

for  $1 \leq i \leq d - k + 1$ . That is, (5.5) holds for  $k = \ell + 1$ ,  $1 \leq i \leq d - k + 1$  and  $n = 1$ . Assume for  $n = m \geq 1$ , (5.5) holds for  $k = \ell + 1$ ,  $1 \leq i \leq d - k + 1$  and  $n = m$ . For  $n = m + 1$ , by (5.6)

$$\begin{aligned}
x_i^k(n) &= x_i^k(m) + \left( \sum_{j=1}^{k-1} x_i^{k-j}(m) \cdot x_{i+k-j}^j \right) + x_i^k \\
&= x_i^k(m) + \left( \sum_{j=1}^{k-1} \left( \sum_{r=1}^{k-j} \binom{m}{r} P_r(\mathbf{x}; i, k-j) \right) \cdot x_{i+k-j}^j \right) + x_i^k \\
&= x_i^k(m) + \left( \sum_{r=1}^{k-1} \left( \sum_{j=1}^{k-r} P_r(\mathbf{x}; i, k-j) x_{i+k-j}^j \binom{m}{r} \right) \right) + x_i^k \\
&= x_i^k(m) + \left( \sum_{r=1}^{k-1} \left( \sum_{j=r}^{k-1} P_r(\mathbf{x}; i, j) x_{i+j}^{k-j} \binom{m}{r} \right) \right) + x_i^k
\end{aligned}$$

for  $1 \leq i \leq d - k + 1$ . Note that

$$\sum_{j=r}^{k-1} P_r(\mathbf{x}; i, j) x_{i+j}^{k-j} = \sum_{j=r}^{k-1} \sum_{\substack{(s_1, \dots, s_r) \in \{1, 2, \dots, k-1\}^r \\ s_1 + \dots + s_r = j}} x_i^{s_1} x_{i+s_1}^{s_2} \dots x_{i+s_1+\dots+s_{r-1}}^{s_r} x_{i+j}^{k-j}$$

which is equal to

$$\sum_{\substack{(s_1, \dots, s_r, s_{r+1}) \in \{1, 2, \dots, k-1\}^{r+1} \\ s_1 + s_2 + \dots + s_r + s_{r+1} = k}} x_i^{s_1} x_{i+s_1}^{s_2} \dots x_{i+s_1+\dots+s_{r-1}}^{s_r} x_{i+s_1+\dots+s_{r-1}+s_r}^{s_{r+1}} = P_{r+1}(\mathbf{x}; i, k)$$

for  $1 \leq r \leq k - 1$  and  $1 \leq i \leq d - k + 1$ . Collecting terms we have

$$\begin{aligned}
x_i^k(n) &= x_i^k(m) + \left( \sum_{r=1}^{k-1} P_{r+1}(\mathbf{x}; i, k) \binom{m}{r} \right) + x_i^k \\
&= x_i^k(m) + \left( \sum_{r=2}^k P_r(\mathbf{x}; i, k) \binom{m}{r-1} \right) + P_1(\mathbf{x}; i, k) \\
&= \left( \sum_{r=1}^m P_r(\mathbf{x}; i, k) \binom{m}{r} \right) + \left( \sum_{r=2}^k P_r(\mathbf{x}; i, k) \binom{m}{r-1} \right) + P_1(\mathbf{x}; i, k).
\end{aligned}$$

Rearranging the order we get

$$\begin{aligned}
x_i^k(n) &= (m+1)P_1(\mathbf{x}; i, k) + \sum_{r=2}^k \left( \binom{m}{r} + \binom{m}{r-1} \right) P_r(\mathbf{x}; i, k) \\
&= \sum_{r=1}^k \binom{m+1}{r} P_r(\mathbf{x}; i, k) = \sum_{r=1}^k \binom{n}{r} P_r(\mathbf{x}; i, k)
\end{aligned}$$

for  $1 \leq i \leq d - k + 1$ . This ends the proof of the lemma.  $\square$



REMARK 5.2.2. By the above lemma, we have

$$P_1(\mathbf{x}; i, k) = x_i^k \text{ and } P_k(\mathbf{x}; i, k) = x_i^1 x_{i+1}^1 \cdots x_{i+k-1}^1$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ .

**5.2.2. Consequences under the inductive assumption.** We will use induction to show Theorem B(2). To make the proof clearer to read, we derive some results under the following inductive assumption.

$$(5.7) \quad \mathcal{F}_{d-1,0} \supset \mathcal{F}_{GP_{d-1}},$$

where  $d \in \mathbb{N}$  with  $d \geq 2$ . For that purpose, we need more notions and lemmas. The proof of Lemma 5.2.3 is similar to the one of Lemma 4.2.5, where  $\mathcal{W}_d$  is defined in Definition 4.2.4.

LEMMA 5.2.3. Under the assumption (5.7), one has for any  $p(n) \in \mathcal{W}_d$  and  $\epsilon > 0$ ,

$$\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{d-1,0}.$$

DEFINITION 5.2.4. For  $r \in \mathbb{N}$ , we define

$$\widetilde{GP}_r = \{p(n) \in GP_r : \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{r,0} \text{ for any } \epsilon > 0\}.$$

REMARK 5.2.5. It is clear that for  $p(n) \in GP_r$ ,  $p(n) \in \widetilde{GP}_r$  if and only if  $-p(n) \in \widetilde{GP}_r$ . Since  $\mathcal{F}_{r,0}$  is a filter, if  $p_1(n), p_2(n), \dots, p_k(n) \in \widetilde{GP}_r$  then

$$p_1(n) + p_2(n) + \dots + p_k(n) \in \widetilde{GP}_r.$$

Moreover by the definition of  $\widetilde{GP}_d$ , we know that  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$  if and only if  $\widetilde{GP}_d = GP_d$ .

LEMMA 5.2.6. Let  $p(n), q(n) \in GP_d$  with  $p(n) \simeq_d q(n)$ . Under the assumption (5.7),  $p(n) \in \widetilde{GP}_d$  if and only if  $q(n) \in \widetilde{GP}_d$ .

PROOF. This follows from Lemma 5.2.3 and the fact that  $\mathcal{F}_{d,0}$  is a filter.  $\square$

For  $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{R}, r \in \mathbb{N}$ , we define  $P(n; \alpha_1, \alpha_2, \dots, \alpha_r)$  as

$$(5.8) \quad \sum_{\ell=1}^r \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = r}} (-1)^{\ell-1} L \left( \frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \frac{n^{j_2}}{j_2!} \prod_{r_2=1}^{j_2} \alpha_{j_1+r_2}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} \alpha_{a(\ell-1)+r_\ell} \right)$$

where the definition of  $L$  is given in (4.1), and  $a(\ell) = \sum_{t=1}^{\ell} j_t$ .

THEOREM 5.2.7. Under the assumption (5.7), the following properties are equivalent:

- (1)  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ .
- (2)  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widetilde{GP}_d$  for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$ , that is
 
$$\{n \in \mathbb{Z} : P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{d,0}$$
 for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$  and  $\epsilon > 0$ .
- (3)  $\text{SGP}_d \subset \widetilde{GP}_d$ .

PROOF. (1)  $\Rightarrow$  (2). Assume  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ . By the definition of  $\widetilde{GP_d}$ , we know that  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$  if and only if  $\widetilde{GP_d} = GP_d$ . Particularly  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widetilde{GP_d}$  for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$ .

(3)  $\Rightarrow$  (1). Assume that  $SGP_d \subset \widetilde{GP_d}$ . Then  $\mathcal{F}_{d,0} \supset \mathcal{F}_{SGP_d}$ . Moreover  $\mathcal{F}_{d,0} \supset \mathcal{F}_{SGP_d} = \mathcal{F}_{GP_d}$  by Theorem 4.2.11.

(2)  $\Rightarrow$  (3). Assume that  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widetilde{GP_d}$  for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$ . We define

$$\Sigma_d = \{(j_1, j_2, \dots, j_\ell) : \ell \in \{1, 2, \dots, d\}, j_1, j_2, \dots, j_\ell \in \mathbb{N} \text{ and } \sum_{t=1}^{\ell} j_t = d\}.$$

For  $(j_1, j_2, \dots, j_\ell), (r_1, r_2, \dots, r_s) \in \Sigma_d$ , we say  $(j_1, j_2, \dots, j_\ell) > (r_1, r_2, \dots, r_s)$  if there exists  $1 \leq t \leq \ell$  such that  $j_t > r_t$  and  $j_i = r_i$  for  $i < t$ . Clearly  $(\Sigma_d, >)$  is a totally ordered set with the maximal element  $(d)$  and the minimal element  $(1, 1, \dots, 1)$ .

For  $\mathbf{j} = (j_1, j_2, \dots, j_\ell) \in \Sigma_d$ , put

$$\mathcal{L}(\mathbf{j}) = \{L(n^{j_1}a_1, \dots, n^{j_\ell}a_\ell) : a_1, \dots, a_\ell \in \mathbb{R}\}.$$

Now, we have

**Claim:**  $\mathcal{L}(\mathbf{s}) \subset \widetilde{GP_d}$  for each  $\mathbf{s} \in \Sigma_d$ .

PROOF. We do induction for  $\mathbf{s}$  under the order  $>$ . First, consider the case when  $\mathbf{s} = (d)$ . Given  $a_1 \in \mathbb{R}$ , we take  $\alpha_1 = 1, \alpha_2 = 2, \dots, \alpha_{d-1} = d-1$  and  $\alpha_d = da_1$ .

Then for any  $1 \leq j_1 \leq d-1$ ,  $\frac{n^{j_1}}{j_1!} \prod_{t=1}^{j_1} \alpha_t \in \mathbb{Z}$  for  $n \in \mathbb{Z}$ . Thus

$$P(n; \alpha_1, \alpha_2, \dots, \alpha_d) = L\left(\frac{n^d}{d!} \prod_{t=1}^d \alpha_t\right) = L(n^d a_1) \pmod{\mathbb{Z}}$$

for any  $n \in \mathbb{Z}$ . Hence  $L(n^d a_1) \in \widetilde{GP_d}$  since  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widetilde{GP_d}$ . Since  $a_1$  is arbitrary, we conclude that  $\mathcal{L}((d)) \subset \widetilde{GP_d}$ .

Assume that for any  $\mathbf{s} > \mathbf{i} = (i_1, \dots, i_k) \in \Sigma_d$ , we have  $\mathcal{L}(\mathbf{s}) \subset \widetilde{GP_d}$ . Now consider the case when  $\mathbf{s} = \mathbf{i} = (i_1, \dots, i_k)$ . There are two cases.

The first case is  $k = d$ ,  $i_1 = i_2 = \dots = i_d = 1$ . Given  $a_1, a_2, \dots, a_d \in \mathbb{R}$ , by the assumption we have that for any  $(j_1, j_2, \dots, j_\ell) > \mathbf{i}$ ,  $\mathcal{L}((j_1, j_2, \dots, j_\ell)) \subset \widetilde{GP_d}$ . Thus

$$\sum_{\ell=1}^{d-1} \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = r}} (-1)^{\ell-1} L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} a_{r_1}, \frac{n^{j_2}}{j_2!} \prod_{r_2=1}^{j_2} a_{j_1+r_2}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} a_{a(\ell-1)+r_\ell}\right)$$

belongs to  $\widetilde{GP_d}$  by the Remark 5.2.5. This implies

$$P(n; a_1, a_2, \dots, a_d) - (-1)^{d-1} L(na_1, na_2, \dots, na_d) \in \widetilde{GP_d}$$

by (5.8). Combining this with  $P(n; a_1, a_2, \dots, a_d) \in \widetilde{GP_d}$ , we have

$$L(na_1, na_2, \dots, na_d) \in \widetilde{GP_d}$$

by Remark 5.2.5. Since  $a_1, a_2, \dots, a_d \in \mathbb{R}$  are arbitrary, we get  $\mathcal{L}(\mathbf{i}) \subset \widetilde{GP}_d$ .

The second case is  $\mathbf{i} > (1, 1, \dots, 1)$ . Given  $a_1, a_2, \dots, a_k \in \mathbb{R}$ , for  $r = 1, 2, \dots, k$ , we put  $\alpha_{\sum_{t=1}^{r-1} i_t + h} = h$  for  $1 \leq h \leq i_r - 1$  and  $\alpha_{\sum_{t=1}^{r-1} i_t + i_r} = i_r a_r$ .

By the assumption, for  $(j_1, j_2, \dots, j_\ell) > \mathbf{i}$ ,

$$L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \frac{n^{j_2}}{j_2!} \prod_{r_2=1}^{j_2} \alpha_{j_1+r_2}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} \alpha_{a(\ell-1)+r_\ell}\right) \in \widetilde{GP}_d.$$

For  $(j_1, j_2, \dots, j_\ell) < \mathbf{i}$ , there exists  $1 \leq u \leq k$  such that  $j_t = i_t$  for  $1 \leq t \leq u-1$  and  $i_u > j_u$ . Then

$$(5.9) \quad \frac{n^{j_u}}{j_u!} \prod_{r_u=1}^{j_u} \alpha_{a(u-1)+r_u} = n^{j_u}.$$

When  $u = 1$ , by (5.9),

$$L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} \alpha_{a(\ell-1)+r_\ell}\right) \in \mathbb{Z}$$

for any  $n \in \mathbb{Z}$ . Hence

$$L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} \alpha_{a(\ell-1)+r_\ell}\right) \in \widetilde{GP}_d.$$

When  $u > 1$ , write  $\beta_v = \frac{1}{j_v!} \prod_{r_v=1}^{j_v} \alpha_{a(v-1)+r_v}$  for  $v = 1, 2, \dots, \ell$ . Then  $\beta_u = 1$  and

$$[L(n^{j_u} \beta_u, n^{j_{u+1}} \beta_{u+1} \dots, n^{j_\ell} \beta_\ell)] = L(n^{j_u} \beta_u, n^{j_{u+1}} \beta_{u+1} \dots, n^{j_\ell} \beta_\ell).$$

Moreover,

$$\begin{aligned} & L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \dots, \frac{n^{j_u}}{j_u!} \prod_{r_u=1}^{j_u} \alpha_{a(u-1)+r_u}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} \alpha_{a(\ell-1)+r_\ell}\right) \\ &= L(n^{j_1} \beta_1, \dots, n^{j_u} \beta_u, \dots, n^{j_\ell} \beta_\ell) \\ &= L(n^{j_1} \beta_1, \dots, n^{j_{u-1}} \beta_{u-1} [L(n^{j_u} \beta_u, \dots, n^{j_\ell} \beta_\ell)]) \end{aligned}$$

which is equal to

$$\begin{aligned} & L(n^{j_1} \beta_1, \dots, n^{j_{u-1}} \beta_{u-1} L(n^{j_u} \beta_u, n^{j_{u+1}} \beta_{u+1} \dots, n^{j_\ell} \beta_\ell)) \\ &= L(n^{j_1} \beta_1, \dots, n^{j_{u-1}+j_u} \beta_{u-1} \beta_u [L(n^{j_{u+1}} \beta_{u+1} \dots, n^{j_\ell} \beta_\ell)]) \\ &= L(n^{j_1} \beta_1, \dots, n^{j_{u-1}+j_u} \beta_{u-1} \beta_u, n^{j_{u+1}} \beta_{u+1} \dots, n^{j_\ell} \beta_\ell) \in \widetilde{GP}_d \end{aligned}$$

since  $(j_1, \dots, j_{u-2}, j_{u-1} + j_u, j_{u+1}, \dots, j_\ell) > \mathbf{i}$ .

Summing up for any  $\mathbf{j} = (j_1, \dots, j_\ell) \in \Sigma_d$  with  $\mathbf{j} \neq \mathbf{i}$ , we have

$$L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \dots, \frac{n^{j_u}}{j_u!} \prod_{r_u=1}^{j_u} \alpha_{a(u-1)+r_u}, \dots, \frac{n^{j_\ell}}{j_\ell!} \prod_{r_\ell=1}^{j_\ell} \alpha_{a(\ell-1)+r_\ell}\right) \in \widetilde{GP}_d.$$

Combining this with  $P(n; \alpha_1, \dots, \alpha_d) \in \widetilde{GP}_d$ , we have

$$\begin{aligned} & L\left(n^{i_1}a_1, n^{i_2}a_2, \dots, n^{i_k}a_k\right) \\ &= L\left(\frac{n^{i_1}}{i_1!} \prod_{r_1=1}^{i_1} \alpha_{r_1}, \frac{n^{i_2}}{i_2!} \prod_{r_2=1}^{i_2} \alpha_{i_1+r_2}, \dots, \frac{n^{i_k}}{i_k!} \prod_{r_k=1}^{i_k} \alpha_{\sum_{t=1}^{k-1} i_t+r_k}\right) \in \widetilde{GP}_d \end{aligned}$$

by (5.8) and Remark (5.2.5). Since  $a_1, \dots, a_k \in \mathbb{R}$  are arbitrary,  $\mathcal{L}(\mathbf{i}) \subset \widetilde{GP}_d$ .  $\square$

Finally, since  $\text{SGP}_d = \bigcup_{\mathbf{j} \in \Sigma_d} \mathcal{L}(\mathbf{j})$ , we have  $\text{SGP}_d \subset \widetilde{GP}_d$  by the above Claim.  $\square$

**5.2.3. Proof of Theorem B(2).** We are now ready to give the proof of the Theorem B(2). As we said before, we will use induction to show Theorem B(2). Firstly, for  $d = 1$ , since  $\mathcal{F}_{GP_1} = \mathcal{F}_{SGP_1}$  and  $\mathcal{F}_{1,0}$  is a filter, it is sufficient to show for any  $a \in \mathbb{R}$  and  $\epsilon > 0$ ,

$$\{n \in \mathbb{Z} : an \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{1,0}.$$

This is obvious since the rotation on the unit circle is a 1-step nilsystem.

Now we assume that  $\mathcal{F}_{d-1,0} \supset \mathcal{F}_{GP_{d-1}}$ , i.e. the assumption (5.7) holds. By Theorem 5.2.7, to show  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ , it remains to prove that  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widetilde{GP}_d$  for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$ , that is

$$\{n \in \mathbb{Z} : P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{d,0}$$

for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$  and  $\epsilon > 0$ .

Let  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$  and choose  $\mathbf{x} = (x_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  with  $x_i^1 = \alpha_i$  for  $i = 1, 2, \dots, d$  and  $x_i^k = 0$  for  $2 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Then

$$A = \mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & \alpha_{d-1} & 0 \\ 0 & 0 & 0 & \dots & 1 & \alpha_d \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

For  $n \in \mathbb{Z}$ , if  $\mathbf{x}(n) = (x_i^k(n))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  satisfies  $\mathbf{M}(\mathbf{x}(n)) = A^n$ , then  $x_i^k(n)$  is a polynomial of  $n$  for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Moreover by Lemma 5.2.1 and Remark 5.2.2, when  $n \in \mathbb{Z}$

$$(5.10) \quad x_i^k(n) = \binom{n}{k} P_k(\mathbf{x}; i, k) = \binom{n}{k} x_i^1 x_{i+1}^1 \dots x_{i+k-1}^1 = \binom{n}{k} \alpha_i \alpha_{i+1} \dots \alpha_{i+k-1}$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ , where  $\binom{n}{k} = \frac{n(n-1) \dots (n-k+1)}{k!}$ .

Now we define  $f_i^1(n) = \lceil x_i^1(n) \rceil = \lceil n\alpha_i \rceil$  for  $1 \leq i \leq d$  and inductively for  $k = 2, 3, \dots, d$  define

$$(5.11) \quad f_i^k(n) = \left\lceil x_i^k(n) - \sum_{j=1}^{k-1} x_i^{k-j}(n) f_{i+k-j}^j(n) \right\rceil$$

for  $1 \leq i \leq d - k + 1$ . Then we define

$$z_i^1(n) = x_i^1(n) - f_i^1(n)$$

for  $1 \leq i \leq d$  and inductively for  $k = 2, 3, \dots, d$  define

$$(5.12) \quad z_i^k(n) = x_i^k(n) - \left( \sum_{j=1}^{k-1} x_i^{k-j}(n) f_{i+k-j}^j(n) \right) - f_i^k(n)$$

for  $1 \leq i \leq d - k + 1$ .

It is clear that  $z_i^k(n) \in GP_k$  for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . First, we have

**Claim:**  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \simeq_d z_1^d(n)$ .

Since the proof of the Claim is long, the readers find the proof in the following subsection. Now we are going to show  $z_1^d(n) \in \widetilde{GP}_d$ .

Let  $X = \mathbb{G}_d/\Gamma$  and  $T$  be the nilrotation induced by  $A \in \mathbb{G}_d$ , i.e.  $B\Gamma \mapsto AB\Gamma$  for  $B \in \mathbb{G}_d$ . Since  $\mathbb{G}_d$  is a  $d$ -step nilpotent Lie group and  $\Gamma$  is a uniform subgroup of  $\mathbb{G}_d$ ,  $(X, T)$  is a  $d$ -step nilsystem.

Let  $I$  be the  $(d+1) \times (d+1)$  identity matrix. For a given  $\eta > 0$ , choose the open neighborhood  $V = \{C \in \mathbb{G}_d : \|C - I\|_\infty < \min\{\frac{1}{2}, \eta\}\}$  of  $I$  in  $\mathbb{G}_d$ . Let  $x_0 = \Gamma \in X$  and  $U = V\Gamma$ . Then  $U$  is an open neighborhood of  $x_0$  in  $X$ . Put

$$S = \{n \in \mathbb{Z} : A^n \Gamma \in U\} = \{n \in \mathbb{Z} : T^n x_0 \in U\}.$$

Then  $S \in \mathcal{F}_{d,0}$ . In the following we are going to show that

$$\{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\} \supset S.$$

This clearly implies that  $\{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\} \in \mathcal{F}_{d,0}$  since  $S \in \mathcal{F}_{d,0}$ . As  $\eta > 0$  is arbitrary, we conclude that  $z_1^d(n) \in \widetilde{GP}_d$ .

Given  $n \in S$ , one has  $A^n \Gamma \in V\Gamma$ . Thus there exists  $B_n \in \Gamma$  such that  $A^n B_n \in V$ , that is,

$$(5.13) \quad \|A^n B_n - I\|_\infty < \min\left\{\frac{1}{2}, \eta\right\}.$$

Take  $\mathbf{h}(n) = (-h_i^k(n))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2}$  with  $\mathbf{M}(\mathbf{h}(n)) = B_n$ . Let  $\mathbf{y}(n) = (y_i^k(n))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  such that

$$\mathbf{M}(\mathbf{y}(n)) = A^n B_n = \mathbf{M}(\mathbf{x}(n))\mathbf{M}(\mathbf{h}(n)).$$

By (3.2)

$$(5.14) \quad y_i^k(n) = x_i^k(n) - \left( \sum_{j=1}^{k-1} x_i^{k-j}(n) h_{i+k-j}^j(n) \right) - h_i^k(n)$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Thus

$$(5.15) \quad |y_i^k(n)| < \min\left\{\frac{1}{2}, \eta\right\}$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$  by (5.13). Hence  $h_i^1(n) = \lceil x_i^1(n) \rceil = \lceil n\alpha_i \rceil$  for  $1 \leq i \leq d$  and

$$(5.16) \quad h_i^k(n) = \left\lceil x_i^k(n) - \sum_{j=1}^{k-1} x_i^{k-j}(n) h_{i+k-j}^j(n) \right\rceil$$

for  $2 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ .

Since  $h_i^1(n) = \lceil n\alpha_i \rceil = f_i^1(n)$  for  $1 \leq i \leq d$ , one has  $h_i^k(n) = f_i^k(n)$  for  $2 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$  by (5.11) and (5.16). Moreover by (5.12) and (5.14), we know  $z_i^k(n) = y_i^k(n)$  for  $2 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Combining this with (5.15),  $|z_i^k(n)| < \min\{\frac{1}{2}, \eta\}$  for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Particularly,  $|z_1^d(n)| < \eta$ . Thus

$$n \in \{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\},$$

which implies that  $\{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\} \supset S$ . That is,  $z_1^d(n) \in \widetilde{GP}_d$ .

Finally using the Claim and the fact that  $z_1^d(n) \in \widetilde{GP}_d$  we have  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widetilde{GP}_d$  by Lemma 5.2.6. This ends the proof, i.e. we have proved  $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ .

#### 5.2.4. Proof of the Claim. Let

$$u_i^k(n) = z_i^k(n) + f_i^k(n) = x_i^k(n) - \sum_{j=1}^{k-1} x_i^{k-j}(n) f_{i+k-j}^j(n)$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Then

$$f_i^k(n) = \lceil u_i^k(n) \rceil$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ .

We define  $U(n; j_1) = \frac{n^{j_1}}{j_1!} \prod_{r=1}^{j_1} \alpha_r$  for  $1 \leq j_1 \leq d$  and recall that  $a(\ell) = \sum_{t=1}^{\ell} j_t$ . Then inductively for  $\ell = 2, 3, \dots, d$  we define

$$\begin{aligned} U(n; j_1, j_2, \dots, j_\ell) &= (U(n; j_1, \dots, j_{\ell-1}) - \lceil U(n; j_1, \dots, j_{\ell-1}) \rceil) \frac{n^{j_\ell}}{j_\ell!} \prod_{r=1}^{j_\ell} \alpha_{a(\ell-1)+r} \\ &= (U(n; j_1, \dots, j_{\ell-1}) - \lceil U(n; j_1, \dots, j_{\ell-1}) \rceil) L\left(\frac{n^{j_\ell}}{j_\ell!} \prod_{r=1}^{j_\ell} \alpha_{a(\ell-1)+r}\right) \end{aligned}$$

for  $j_1, j_2, \dots, j_\ell \geq 1$  and  $j_1 + \dots + j_\ell \leq d$  (see (4.1) for the definition of  $L$ ).

Next,  $U(n; d) = \frac{n^d}{d!} \prod_{r=1}^d \alpha_r = L(\frac{n^d}{d!} \prod_{r=1}^d \alpha_r)$  and for  $2 \leq \ell \leq d$ ,  $j_1, j_2, \dots, j_\ell \in \mathbb{N}$  with  $j_1 + j_2 + \dots + j_\ell = d$ , by Lemma 4.2.7(1)

$$\begin{aligned} U(n; j_1, j_2, \dots, j_\ell) &= (U(n; j_1, \dots, j_{\ell-1}) - \lceil U(n; j_1, \dots, j_{\ell-1}) \rceil) L\left(\frac{n^{j_\ell}}{j_\ell!} \prod_{r=1}^{j_\ell} \alpha_{a(\ell-1)+r}\right) \\ &\simeq_d U(n; j_1, \dots, j_{\ell-1}) \lceil L\left(\frac{n^{j_\ell}}{j_\ell!} \prod_{r=1}^{j_\ell} \alpha_{a(\ell-1)+r}\right) \rceil \end{aligned}$$

which is equal to

$$\begin{aligned} & (U(n; j_1, \dots, j_{\ell-2}) - \lceil U(n; j_1, \dots, j_{\ell-2}) \rceil) \times L\left(\frac{n^{j_{\ell-1}}}{j_{\ell-1}!} \prod_{r_{\ell-1}=1}^{j_{\ell-1}} \alpha_{a(\ell-2)+r_{\ell-1}}, \frac{n^{j_{\ell}}}{j_{\ell}!} \prod_{r_{\ell}=1}^{j_{\ell}} \alpha_{a(\ell-1)+r_{\ell}}\right) \\ & \simeq_d U(n; j_1, \dots, j_{\ell-2}) \lceil L\left(\frac{n^{j_{\ell-1}}}{j_{\ell-1}!} \prod_{r_{\ell-1}=1}^{j_{\ell-1}} \alpha_{a(\ell-2)+r_{\ell-1}}, \frac{n^{j_{\ell}}}{j_{\ell}!} \prod_{r_{\ell}=1}^{j_{\ell}} \alpha_{a(\ell-1)+r_{\ell}}\right) \rceil. \end{aligned}$$

Continuing the above argument we have

$$U(n; j_1, j_2, \dots, j_{\ell}) \simeq_d L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \frac{n^{j_2}}{j_2!} \prod_{r_2=1}^{j_2} \alpha_{j_1+r_2}, \dots, \frac{n^{j_{\ell}}}{j_{\ell}!} \prod_{r_{\ell}=1}^{j_{\ell}} \alpha_{a(\ell-1)+r_{\ell}}\right).$$

That is, for  $1 \leq \ell \leq d$ ,  $j_1, j_2, \dots, j_{\ell} \in \mathbb{N}$  with  $j_1 + j_2 + \dots + j_{\ell} = d$ ,

(5.17)

$$U(n; j_1, j_2, \dots, j_{\ell}) \simeq_d L\left(\frac{n^{j_1}}{j_1!} \prod_{r_1=1}^{j_1} \alpha_{r_1}, \frac{n^{j_2}}{j_2!} \prod_{r_2=1}^{j_2} \alpha_{j_1+r_2}, \dots, \frac{n^{j_{\ell}}}{j_{\ell}!} \prod_{r_{\ell}=1}^{j_{\ell}} \alpha_{a(\ell-1)+r_{\ell}}\right).$$

Thus using (5.17) we have

$$(5.18) \quad P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \simeq_d \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_{\ell} \in \mathbb{N} \\ j_1 + \dots + j_{\ell} = d}} (-1)^{\ell-1} U(n; j_1, j_2, \dots, j_{\ell}).$$

Next using Lemma 4.2.7(1), for any  $j_1, \dots, j_{\ell} \in \mathbb{N}$  with  $a(\ell) \leq d-1$ , we have

$$\begin{aligned} & U(n; j_1, \dots, j_{\ell}) f_{1+a(\ell)}^{d-a(\ell)}(n) = U(n; j_1, \dots, j_{\ell}) \lceil u_{1+a(\ell)}^{d-a(\ell)}(n) \rceil \\ & \simeq_d \left( U(n; j_1, \dots, j_{\ell}) - \lceil U(n; j_1, \dots, j_{\ell}) \rceil \right) u_{1+a(\ell)}^{d-a(\ell)}(n) \\ & = \left( U(n; j_1, \dots, j_{\ell}) - \lceil U(n; j_1, \dots, j_{\ell}) \rceil \right) \times \left( x_{1+a(\ell)}^{d-a(\ell)}(n) - \sum_{j_{\ell+1}=1}^{d-a(\ell)-1} x_{1+a(\ell)}^{j_{\ell+1}}(n) f_{1+a(\ell+1)}^{d-a(\ell+1)}(n) \right) \\ & = \left( U(n; j_1, j_2, \dots, j_{\ell}) - \lceil U(n; j_1, j_2, \dots, j_{\ell}) \rceil \right) \times \\ & \quad \left( \binom{n}{d-a(\ell)} \prod_{r_{\ell+1}=1}^{d-a(\ell)} \alpha_{a(\ell)+r_{\ell+1}} - \sum_{j_{\ell+1}=1}^{d-a(\ell)-1} \binom{n}{j_{\ell+1}} \prod_{r_{\ell+1}=1}^{j_{\ell+1}} \alpha_{a(\ell)+r_{\ell+1}} f_{1+a(\ell+1)}^{d-a(\ell+1)}(n) \right) \\ & \simeq_d \left( U(n; j_1, j_2, \dots, j_{\ell}) - \lceil U(n; j_1, j_2, \dots, j_{\ell}) \rceil \right) \times \\ & \quad \left( \frac{n^{d-a(\ell)}}{(d-a(\ell))!} \prod_{r_{\ell+1}=1}^{d-a(\ell)} \alpha_{a(\ell)+r_{\ell+1}} - \sum_{j_{\ell+1}=1}^{d-a(\ell)-1} \frac{n^{j_{\ell+1}}}{j_{\ell+1}!} \prod_{r_{\ell+1}=1}^{j_{\ell+1}} \alpha_{a(\ell)+r_{\ell+1}} f_{1+a(\ell+1)}^{d-a(\ell+1)}(n) \right) \\ & = U(n; j_1, \dots, j_{\ell}, d-a(\ell)) - \sum_{j_{\ell+1}=1}^{d-a(\ell)-1} U(n; j_1, \dots, j_{\ell}, j_{\ell+1}) f_{1+a(\ell+1)}^{d-a(\ell+1)}(n). \end{aligned}$$

Using this fact and Lemma 4.2.7(1), we have

$$\begin{aligned}
z_1^d(n) &\simeq_d u_1^d(n) = x_1^d(n) - \sum_{j_1=1}^{d-1} x_1^{j_1}(n) f_{1+j_1}^{d-j_1}(n) \\
&= \binom{n}{d} \alpha_1 \alpha_2 \dots \alpha_d - \sum_{j_1=1}^{d-1} \binom{n}{j_1} \alpha_1 \alpha_2 \dots \alpha_{j_1} f_{1+j_1}^{d-j_1}(n) \simeq_d U(n; d) - \sum_{j_1=1}^{d-1} U(n; j_1) f_{1+j_1}^{d-j_1}(n) \\
&\simeq_d U(n; d) - \left( \sum_{j_1=1}^{d-1} (U(n; j_1, d-j_1) - \sum_{j_2=1}^{d-j_1-1} U(n; j_1, j_2) f_{1+j_1+j_2}^{d-(j_1+j_2)}(n)) \right).
\end{aligned}$$

Continuing this argument we obtain

$$z_1^d(n) \simeq_d \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} U(n; j_1, \dots, j_\ell).$$

Combining this with (5.18), we have proved the Claim.



## CHAPTER 6

### Generalized polynomials and recurrence sets: Proof of Theorem C

In this chapter we will prove Theorem C. That is, we will show that for  $d \in \mathbb{N}$  and  $F \in \mathcal{F}_{GP_d}$ , there exist a minimal  $d$ -step nilsystem  $(X, T)$  and a nonempty open set  $U$  such that

$$F \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

Let us explain the idea of the proof of Theorem C. Put

$$(6.1) \quad \mathcal{N}_d = \{B \subset \mathbb{Z} : \text{there are a minimal } d\text{-step nilsystem } (X, T) \text{ and an open non-empty set } U \text{ of } X \text{ with } B \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}\}.$$

Similar to the proof of Theorem B(2) we first show that  $\mathcal{F}_{GP_d} \subset \mathcal{N}_d$  if and only if  $\{n \in \mathbb{Z} : \|P(n; \alpha_1, \dots, \alpha_d)\| < \epsilon\} \in \mathcal{N}_d$  for any  $\alpha_1, \dots, \alpha_d \in \mathbb{R}$  and  $\epsilon > 0$ . We choose  $(X, T)$  as the closure of the orbit of  $\Gamma$  in  $\mathbb{G}_d/\Gamma$  (the nilrotation is induced by a matrix  $A \in \mathbb{G}_d$ ), define  $U \subset X$  depending on a given  $\epsilon > 0$ , put  $S = \{n \in \mathbb{Z} : \bigcap_{i=0}^d T^{-in}U \neq \emptyset\}$ ; and consider the most right-corner entry  $z_1^d(m)$  in  $A^{nm}BC_m$  with  $B \in \mathbb{G}_d$  and  $C_m \in \Gamma$  for a given  $n \in S$  with  $1 \leq m \leq d$ . We finish the proof by showing  $S \subset \{n \in \mathbb{Z} : \|P(n; \alpha_1, \dots, \alpha_d)\| < \epsilon\}$  which implies that  $\{n \in \mathbb{Z} : \|P(n; \alpha_1, \dots, \alpha_d)\| < \epsilon\} \in \mathcal{N}_d$ .

#### 6.1. A special case and preparation

**6.1.1. The ordinary polynomial case.** To illustrate the idea of the proof of Theorem C, we first consider the situation when the generalized polynomials are the ordinary ones. That is, we want to explain if  $p(n)$  is a polynomial of degree  $d$  with  $p(0) = 0$  and  $\epsilon > 0$ , how we can find a  $d$ -step nilsystem  $(X, T)$ , and a nonempty open set  $U \subset X$  such that

$$(6.2) \quad \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

To do this define  $T_{\alpha,d} : \mathbb{T}^d \longrightarrow \mathbb{T}^d$  by

$$T_{\alpha,d}(\theta_1, \theta_2, \dots, \theta_d) = (\theta_1 + \alpha, \theta_2 + \theta_1, \theta_3 + \theta_2, \dots, \theta_d + \theta_{d-1}),$$

where  $\alpha \in \mathbb{R}$ . A simple computation yields that

$$(6.3) \quad T_{\alpha,d}^n(\theta_1, \dots, \theta_d) = (\theta_1 + n\alpha, n\theta_1 + \theta_2 + \frac{1}{2}n(n-1)\alpha, \dots, \sum_{i=0}^d \binom{n}{d-i} \theta_i),$$

where  $\theta_0 = \alpha$ ,  $n \in \mathbb{Z}$  and  $\binom{n}{0} = 1$ ,  $\binom{n}{i} := \frac{\prod_{j=0}^{i-1} (n-j)}{i!}$  for  $i = 1, 2, \dots, d$ .

We now prove (6.2) by induction. The case when  $d = 1$  is easy, and we assume that for each polynomial of degree  $\leq d - 1$  (6.2) holds. Now let  $p(n) = \sum_{i=1}^d \alpha_i n^i$  with  $\alpha_i \in \mathbb{R}$ . By induction for each  $1 \leq i \leq d - 1$  there is an  $i$ -step nilsystem  $(X_i, T_i)$  and an open non-empty subset  $U_i$  of  $X_i$  such that

$$\{n \in \mathbb{Z} : \alpha_i n^i \pmod{\mathbb{Z}} \in (-\frac{\epsilon}{d}, \frac{\epsilon}{d})\} \supset \{n \in \mathbb{Z} : U_i \cap T_i^{-n} U_i \cap \dots \cap T_i^{-dn} U_i \neq \emptyset\}.$$

By the Vandermonde's formula, we know

$$\begin{pmatrix} 1 & 2 & 3 & \dots & d \\ 1 & 2^2 & 3^2 & \dots & d^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{d-1} & 3^{d-1} & \dots & d^{d-1} \\ 1 & 2^d & 3^d & \dots & d^d \end{pmatrix}$$

is a non-singular matrix. Hence there are integers  $\lambda_1, \lambda_2, \dots, \lambda_d$  and  $\lambda \in \mathbb{N}$  such that the following equation holds:

$$\begin{pmatrix} 1 & 2 & 3 & \dots & d \\ 1 & 2^2 & 3^2 & \dots & d^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{d-1} & 3^{d-1} & \dots & d^{d-1} \\ 1 & 2^d & 3^d & \dots & d^d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{d-1} \\ \lambda_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda \end{pmatrix}.$$

That is,

$$(6.4) \quad \begin{aligned} \sum_{m=1}^d \lambda_m m^j &= \lambda_1 + \lambda_2 2^j + \dots + \lambda_d d^j = 0, \quad 1 \leq j \leq d-1; \\ \sum_{m=1}^d \lambda_m m^d &= \lambda_1 + \lambda_2 2^d + \dots + \lambda_d d^d = \lambda. \end{aligned}$$

Now let  $T_d = T_{\frac{\alpha_d}{\lambda}, d}$  and  $Y_d = \mathbb{T}^d$ . Let  $K_d = d! \sum_{i=1}^d |\lambda_i|$ ,  $\epsilon_1 > 0$  with  $2K_d \epsilon_1 < \epsilon/d$  and  $U_d = (-\epsilon_1, \epsilon_1)^d$ .

It is easy to see that if  $n \in \{n \in \mathbb{Z} : U_d \cap T_d^{-n} U_d \cap \dots \cap T_d^{-dn} U_d \neq \emptyset\}$  then we know that there is  $(\theta_1, \dots, \theta_d) \in U_d$  such that  $T_d^{in}(\theta_1, \dots, \theta_d) \in U_d$  for each  $1 \leq i \leq d$ . Thus, by (6.3) considering the last coordinate we get that

$$\begin{aligned} \binom{n}{d} \theta_0 + \binom{n}{d-1} \theta_1 + \dots + \binom{n}{0} \theta_d \pmod{\mathbb{Z}} &\in (-\epsilon_1, \epsilon_1) \\ \binom{2n}{d} \theta_0 + \binom{2n}{d-1} \theta_1 + \dots + \binom{2n}{0} \theta_d \pmod{\mathbb{Z}} &\in (-\epsilon_1, \epsilon_1) \\ &\dots \dots \dots \\ \binom{dn}{d} \theta_0 + \binom{dn}{d-1} \theta_1 + \dots + \binom{dn}{0} \theta_d \pmod{\mathbb{Z}} &\in (-\epsilon_1, \epsilon_1), \end{aligned}$$

where  $\theta_0 = \frac{\alpha_d}{\lambda}$ . Multiplying  $\binom{in}{d} \theta_0 + \binom{in}{d-1} \theta_1 + \dots + \binom{in}{0} \theta_d$  by  $\lambda_i d!$  and summing over  $i = 1, \dots, d$  we get that

$$\sum_{j=1}^d \lambda_j d! \sum_{i=0}^d \binom{jn}{d-i} \theta_i = \theta_d \left( \sum_j \lambda_j \right) d! + \alpha_d n^d \pmod{\mathbb{Z}} \in (-K_d \epsilon_1, K_d \epsilon_1).$$

Thus  $\alpha_d n^d \pmod{\mathbb{Z}} \in (-2K_d \epsilon_1, 2K_d \epsilon_1) \subset (-\epsilon/d, \epsilon/d)$ .

Choose  $x_i \in U_i$  for  $1 \leq i \leq d$ . Let  $x = (x_1, x_2, \dots, x_d) \in X_1 \times \dots \times X_d$  and  $X$  be the orbit closure of  $x$  under  $T = T_1 \times T_2 \dots \times T_d$ . Then  $(X, T)$  is a  $d$ -step nilsystem. If we let  $U = (U_1 \times U_2 \times \dots \times U_d) \cap X$ , then we have (6.2).

By the property of nilsystems and the discussion above it is easy to see

REMARK 6.1.1. Let  $k \in \mathbb{N}$ ,  $q_i(x)$  be a polynomial of degree  $d$  with  $q_i(0) = 0$  and  $\epsilon_i > 0$  for  $1 \leq i \leq k$ . Then there are a  $d$ -step nilsystem  $(X, T, \mu)$  and  $B \subset X$  with  $\mu(B) > 0$  such that

$$\bigcap_{i=1}^k \{n \in \mathbb{Z} : \|q_i(n)\| < \epsilon_i\} \supset \{n \in \mathbb{Z} : \mu(B \cap T^{-n}B \cap \dots \cap T^{-dn}B) > 0\}$$

**6.1.2. Some preparation.** Recall that for  $d \in \mathbb{N}$ ,  $\mathcal{N}_d$  is defined in (6.1). Hence Theorem C is equivalent to

$$\mathcal{F}_{GP_d} \subset \mathcal{N}_d.$$

LEMMA 6.1.2. For each  $d \in \mathbb{N}$ ,  $\mathcal{N}_d$  is a filter.

PROOF. Let  $B_1, B_2 \in \mathcal{N}_d$ . To show  $\mathcal{N}_d$  is a filter, it suffices to show  $B_1 \cap B_2 \in \mathcal{N}_d$ . By definition, there exist minimal  $d$ -step nilsystems  $(X_i, T_i)$ , and nonempty open sets  $U_i$  for  $i = 1, 2$  such that

$$B_i \supset \{n \in \mathbb{Z} : U_i \cap T_i^{-n}U_i \cap \dots \cap T_i^{-dn}U_i \neq \emptyset\}.$$

Taking any minimal point  $x = (x_1, x_2) \in X_1 \times X_2$ , let  $X = \overline{\mathcal{O}(x, T)}$ , where  $T = T_1 \times T_2$ . Note that  $(X, T)$  is also a minimal  $d$ -step nilsystem.

Since  $(X_i, T_i)$ ,  $i = 1, 2$ , are minimal, there are  $k_i \in \mathbb{N}$  such that  $x_i \in T_i^{-k_i}U_i$ ,  $i = 1, 2$ . Let  $U = (T_1^{-k_1}U_1 \times T_2^{-k_2}U_2) \cap X$ , then  $U$  is an open set of  $X$ . Note that

$$\begin{aligned} & \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\} \\ &= \bigcap_{i=1,2} \{n \in \mathbb{Z} : T_i^{-k_i}U_i \cap T_i^{-(k_i+n)}U_i \cap \dots \cap T_i^{-(k_i+dn)}U_i \neq \emptyset\} \\ &= \bigcap_{i=1,2} \{n \in \mathbb{Z} : U_i \cap T_i^{-n}U_i \cap \dots \cap T_i^{-dn}U_i \neq \emptyset\} \end{aligned}$$

Hence

$$B_1 \cap B_2 \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

That is,  $B_1 \cap B_2 \in \mathcal{N}_d$  and  $\mathcal{N}_d$  is a filter.  $\square$

DEFINITION 6.1.3. For  $r \in \mathbb{N}$ , define

$$\widehat{GP}_r = \{p(n) \in GP_r : \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{N}_r, \forall \epsilon > 0\}.$$

REMARK 6.1.4. It is clear that for  $p(n) \in GP_r$ ,  $p(n) \in \widehat{GP}_r$  if and only if  $-p(n) \in \widehat{GP}_r$ . Since  $\mathcal{N}_r$  is a filter, if  $p_1(n), p_2(n), \dots, p_k(n) \in \widehat{GP}_r$  then

$$p_1(n) + p_2(n) + \dots + p_k(n) \in \widehat{GP}_r.$$

Moreover by the definition of  $\widehat{GP}_r$ , we know that  $\mathcal{F}_{GP_r} \subset \mathcal{N}_r$  if and only if  $\widehat{GP}_r = GP_r$ .

We shall prove Theorem C inductively, thus we need to obtain some results under the following assumption, that is for some  $d \geq 2$ ,

$$(6.5) \quad \mathcal{F}_{GP_{d-1}} \subset \mathcal{N}_{d-1}.$$

LEMMA 6.1.5. Let  $p(n), q(n) \in GP_d$  with  $p(n) \simeq_d q(n)$ . Under the assumption (6.5),  $p(n) \in \widehat{GP}_d$  if and only if  $q(n) \in \widehat{GP}_d$ .

PROOF. It follows from Lemma 5.2.3,  $\mathcal{N}_d$  being a filter and  $\mathcal{F}_{GP_{d-1}} \subset \mathcal{N}_{d-1} \subset \mathcal{N}_d$ .  $\square$

THEOREM 6.1.6. Under the assumption (6.5), the following properties are equivalent:

- (1)  $\mathcal{F}_{GP_d} \subset \mathcal{N}_d$ .
- (2)  $P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \in \widehat{GP}_d$  for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$ , that is
 
$$\{n \in \mathbb{Z} : P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{N}_d$$
 for any  $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$  and  $\epsilon > 0$ .
- (3)  $SGP_d \subset \widehat{GP}_d$ .

PROOF. The proof is similar to that of Theorem 5.2.7.  $\square$

## 6.2. Proof of Theorem C

Now we prove  $\mathcal{F}_{GP_d} \subset \mathcal{N}_d$  by induction on  $d$ . When  $d = 1$ , since  $\mathcal{F}_{GP_1} = \mathcal{F}_{SGP_1}$  and  $\mathcal{N}_d$  is a filter, it is sufficient to show that: for any  $p(n) = an \in SGP_1$  and  $\epsilon > 0$ , we have

$$(6.6) \quad \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{N}_1.$$

This is easy to be verified.

Now we assume that for  $d \geq 2$ ,  $\mathcal{F}_{GP_{d-1}} \subset \mathcal{N}_{d-1}$ , i.e. (6.5) holds. Then it follows from Theorem 6.1.6 that under the assumption (6.5), to show  $\mathcal{F}_{GP_d} \subset \mathcal{N}_d$ , it is sufficient to show that

$$P(n; \beta_1, \beta_2, \dots, \beta_d) \in \widehat{GP}_d,$$

for any  $\beta_1, \beta_2, \dots, \beta_d \in \mathbb{R}$ .

Fix  $\beta_1, \beta_2, \dots, \beta_d \in \mathbb{R}$ . We divide the remainder of the proof into two steps.

**Step 1.** We are going to show

$$P(n; \beta_1, \beta_2, \dots, \beta_d) \simeq_d \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell),$$

where as in the proof of Theorem B, we define

$$(6.7) \quad U(n; j_1) = \frac{n^{j_1}}{j_1!} \prod_{r=1}^{j_1} \alpha_r, \quad 1 \leq j_1 \leq d.$$

And inductively for  $\ell = 2, 3, \dots, d$  define

$$\begin{aligned} U(n; j_1, j_2, \dots, j_\ell) &= (U(n; j_1, \dots, j_{\ell-1}) - \lceil U(n; j_1, \dots, j_{\ell-1}) \rceil) \frac{n^{j_\ell}}{j_\ell!} \prod_{r=1}^{j_\ell} \alpha_{\sum_{t=1}^{\ell-1} j_t + r} \\ &= (U(n; j_1, \dots, j_{\ell-1}) - \lceil U(n; j_1, \dots, j_{\ell-1}) \rceil) L\left(\frac{n^{j_\ell}}{j_\ell!} \prod_{r=1}^{j_\ell} \alpha_{\sum_{t=1}^{\ell-1} j_t + r_\ell}\right) \end{aligned}$$

for  $j_1, j_2, \dots, j_\ell \geq 1$  and  $j_1 + \dots + j_\ell \leq d$  (see (4.1) for the definition of  $L$ ).

In fact, let  $\lambda_1, \lambda_2, \dots, \lambda_d \in \mathbb{Z}$  and  $\lambda \in \mathbb{N}$  satisfying (6.4). Put

$$\alpha_1 = \beta_1/\lambda, \alpha_2 = \beta_2, \alpha_3 = \beta_3, \dots, \alpha_d = \beta_d.$$

Then

$$P(n; \beta_1, \beta_2, \dots, \beta_d) = \lambda P(n; \alpha_1, \alpha_2, \dots, \alpha_d).$$

Note that in proof of Theorem B we have

$$(6.8) \quad P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \simeq_d \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} U(n; j_1, j_2, \dots, j_\ell).$$

Since  $\lambda$  is an integer, we have

$$\lambda P(n; \alpha_1, \alpha_2, \dots, \alpha_d) \simeq_d \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell).$$

That is,

$$P(n; \beta_1, \beta_2, \dots, \beta_d) \simeq_d \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell).$$

Hence, by Lemma 6.1.5, to show  $P(n; \beta_1, \beta_2, \dots, \beta_d) \in \widehat{GP}_d$ , it suffices to show

$$(6.9) \quad \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell) \in \widehat{GP}_d.$$

Now choose  $\mathbf{x} = (x_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  with  $x_i^1 = \alpha_i$  for  $i = 1, 2, \dots, d$  and  $x_i^k = 0$  for  $2 \leq k \leq d$  and  $1 \leq i \leq d - k + 1$ . Let

$$A = \mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{d-1} & 0 \\ 0 & 0 & 0 & \dots & 1 & \alpha_d \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

For  $n \in \mathbb{Z}$ , if  $\mathbf{x}(n) = (x_i^k(n))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  satisfies  $\mathbf{M}(\mathbf{x}(n)) = A^n$ , then  $x_i^k(n)$  is a polynomial of  $n$  for  $1 \leq k \leq d$  and  $1 \leq i \leq d-k+1$ . Moreover, by Lemma 5.2.1 and Remark 5.2.2, when  $n \in \mathbb{Z}$

$$(6.10) \quad x_i^k(n) = \binom{n}{k} \alpha_i \alpha_{i+1} \dots \alpha_{i+k-1}$$

for  $1 \leq k \leq d$  and  $1 \leq i \leq d-k+1$ .

Let  $X = \mathbb{G}_d/\Gamma$  and  $T$  be the nilrotation induced by  $A \in \mathbb{G}_d$ , i.e.  $B\Gamma \mapsto AB\Gamma$  for  $B \in \mathbb{G}_d$ . Since  $\mathbb{G}_d$  is a  $d$ -step nilpotent Lie group and  $\Gamma$  is a uniform subgroup of  $\mathbb{G}_d$ ,  $(X, T)$  is a  $d$ -step nilsystem. Let  $x_0 = \Gamma \in X$  and  $Z$  be the closure of the orbit  $\mathcal{O}(x_0, T)$  of  $x_0$  in  $X$ . Then  $(Z, T)$  is a minimal  $d$ -step nilsystem.

**Step 2.** For any  $\epsilon > 0$ , we are going to show there is a nonempty open subset  $U$  of  $Z$  such that

$$(6.11) \quad \begin{aligned} & \{n \in \mathbb{Z} : \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \\ & \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}. \end{aligned}$$

That means  $\sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell) \in \widehat{GP}_d$ .

Fix an  $\epsilon > 0$ . Take  $\epsilon_1 = \min\{\frac{\epsilon}{2K(\sum_{i=0}^{d-1} d^i)}, \frac{1}{4}\}$ , where  $K = \sum_{m=1}^d |\lambda_m| \left(\sum_{t=0}^d m^t\right)$ , and let  $V = \{C \in \mathbb{G}_d : \|C - I\|_\infty < \epsilon_1\}$  be a neighborhood of  $I$  in  $\mathbb{G}_d$ . Put  $U = V\Gamma \cap Z$ . Then  $U$  is an open neighborhood of  $x_0$  in  $Z$ . Let

$$S = \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset\}.$$

Now we show that

$$S \subset \left\{n \in \mathbb{Z} : \sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\right\}.$$

Let  $n \in S$ . Then  $U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset$ . Hence there is some  $B \in \mathbb{G}_d$  with

$$B\Gamma \in U \cap T^{-n}U \cap \dots \cap T^{-dn}U.$$

Thus  $A^{mn}B\Gamma \in V\Gamma$ ,  $m = 0, 1, 2, \dots, d$ . We may assume that  $B \in V$ .

For each  $m \in \{1, 2, \dots, d\}$ , since  $A^{mn}B\Gamma \in V\Gamma$  there is some  $C_m \in \Gamma$  such that

$$(6.12) \quad \|A^{mn}BC_m - I\|_\infty < \epsilon_1.$$

Let  $A^{mn}BC_m = \mathbf{M}(\mathbf{z}(m))$ , where  $\mathbf{z}(m) = (z_i^k(m))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$ . Then from (6.12), we have

$$|z_i^k(m)| < \epsilon_1, \quad 1 \leq k \leq d, 1 \leq i \leq d-k+1.$$

On the one hand, since  $|z_1^d(m)| < \epsilon_1$ , we have

$$(6.13) \quad \sum_{m=1}^d \lambda_m z_1^d(m) \in (-K\epsilon_1, K\epsilon_1).$$

On the other hand, we have

$$(6.14) \quad \begin{aligned} \sum_{m=1}^d \lambda_m z_1^d(m) &\approx \left( \sum_{l=1}^d (-1)^{l-1} \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1 + j_2 + \dots + j_l = d}} \lambda U(n; j_1, j_2, \dots, j_l) \right) \\ &\quad + \Delta \left( (d + d^2 + \dots + d^{d-1})(2K\epsilon_1) \right). \end{aligned}$$

Note that for  $a, b \in \mathbb{R}$  and  $\delta > 0$ ,  $a \approx b + \Delta(\delta)$  means that  $a - b \pmod{\mathbb{Z}} \in (-\delta, \delta)$ .

Since the proof of (6.14) is long, we put it after Theorem C. Now we continue the proof. By (6.14) and (6.13), we have

$$\sum_{l=1}^d (-1)^{l-1} \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1 + j_2 + \dots + j_l = d}} \lambda U(n; j_1, j_2, \dots, j_l) \pmod{\mathbb{Z}} \in \left( -M(2K\epsilon_1), M(2K\epsilon_1) \right) \subset (-\epsilon, \epsilon),$$

where  $M = 1 + d + \dots + d^{d-1}$ . This means that

$$n \in \left\{ q \in \mathbb{Z} : \sum_{l=1}^d \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1 + j_2 + \dots + j_l = d}} (-1)^{l-1} \lambda U(q; j_1, j_2, \dots, j_l) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon) \right\}.$$

Hence

$$S \subset \left\{ q \in \mathbb{Z} : \sum_{l=1}^d \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1 + j_2 + \dots + j_l = d}} (-1)^{l-1} \lambda U(q; j_1, j_2, \dots, j_l) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon) \right\}.$$

Thus we have proved (6.11) which means  $\sum_{\ell=1}^d \sum_{\substack{j_1, \dots, j_\ell \in \mathbb{N} \\ j_1 + \dots + j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \dots, j_\ell) \in$

$\widehat{GP}_d$ . The proof of Theorem C is now finished.

**6.2.1. Proof of (6.14).** Since  $B \in V$ ,

$$(6.15) \quad \|B - I\|_\infty < \epsilon_1 < 1/2.$$

Denote  $B = \mathbf{M}(\mathbf{y})$ , where  $\mathbf{y} = (y_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$ . From (6.15),

$$|y_i^k| < \epsilon_1, \quad 1 \leq k \leq d, \quad 1 \leq i \leq d - k + 1.$$

For  $m = 1, 2, \dots, d$ , recall that  $C_m \in \Gamma$  satisfies (6.12). Denote  $C_m = \mathbf{M}(\mathbf{h}(m))$ , where  $\mathbf{h}(m) = (-h_i^k(m))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2}$ . Let  $A^{mn}B = \mathbf{M}(\mathbf{w}(m))$ , where

$\mathbf{w}(m) = (w_i^k(m))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$ . Then

$$\begin{aligned}
 w_i^k(m) &= x_i^k(mn) + \left( \sum_{j=1}^{k-1} x_i^j(mn) y_{i+j}^{k-j} \right) + y_i^k \\
 (6.16) \quad &= \binom{mn}{k} \alpha_i \alpha_{i+1} \dots \alpha_{i+k-1} + \sum_{j=1}^{k-1} \binom{mn}{j} \alpha_i \alpha_{i+1} \dots \alpha_{i+j-1} y_{i+j}^{k-j} + y_i^k \\
 &\triangleq \frac{(mn)^k}{k!} \alpha_i \dots \alpha_{i+k-1} + \sum_{j=1}^{k-1} m^j a_i^k(j) + a_i^k(0),
 \end{aligned}$$

where  $m = 1, 2, \dots, d$ ,  $a_i^k(j)$  does not depend on  $m$  and  $|a_i^k(0)| = |y_i^k| < \epsilon_1$ .

Recall that  $\mathbf{z}(m) = (z_i^k(m))_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{R}^{d(d+1)/2}$  satisfies  $A^{mn}BC_m = \mathbf{M}(\mathbf{z}(m))$ . Hence

$$(6.17) \quad z_i^k(m) = w_i^k(m) - \left( \sum_{j=1}^{k-1} w_i^j(m) h_{i+j}^{k-j}(m) \right) - h_i^k(m).$$

From  $\|A^{mn}BC_m - I\|_\infty < \epsilon_1$ , we have

$$|z_i^k(m)| < \epsilon_1, \quad 1 \leq k \leq d, 1 \leq i \leq d - k + 1.$$

Note that  $h_i^k(m) \in \mathbb{Z}$ , and we have

$$h_i^k(m) = \left\lceil w_i^k(m) - \sum_{j=1}^{k-1} w_i^j(m) h_{i+j}^{k-j}(m) \right\rceil.$$

Let

$$u_i^k(m) = w_i^k(m) - \sum_{j=1}^{k-1} w_i^j(m) h_{i+j}^{k-j}(m).$$

Then

$$|u_i^k(m) - h_i^k(m)| = |z_i^k(m)| < \epsilon_1 < 1/2.$$

Recall that for  $a, b \in \mathbb{R}$  and  $\delta > 0$ ,  $a \approx b + \Delta(\delta)$  means  $a - b \pmod{\mathbb{Z}} \in (-\delta, \delta)$ .

**Claim:** Let  $1 \leq r \leq d-1$  and  $v_r(0), v_r(1), \dots, v_r(r) \in \mathbb{R}$ . Then for each  $1 \leq r_1 \leq d-r-1$  and  $1 \leq j \leq r_1+r$ , there exist  $v_{r,r_1}(j) \in \mathbb{R}$  such that

(1) we have

$$\begin{aligned}
 \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t v_r(t) \right) h_{1+r}^{d-r}(m) &\approx \lambda(v_r(r) - \lceil v_r(r) \rceil) \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d \\
 &- \sum_{r_1=1}^{d-r-1} \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^{r_1+r} m^t v_{r,r_1}(t) \right) h_{1+r+r_1}^{d-r-r_1}(m) + \Delta(2K\epsilon_1)
 \end{aligned}$$

(2)  $v_{r,r_1}(r+r_1) = \left( v_r(r) - \lceil v_r(r) \rceil \right) \frac{n^{r_1}}{r_1!} \alpha_{r+1} \dots \alpha_{r+r_1}$  for all  $1 \leq r_1 \leq d-r-1$ .



PROOF OF CLAIM. First we have

$$\left| \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(r) - \lceil v_r(r) \rceil) \right) \right| \leq \sum_{m=1}^d |\lambda_m| \left( \sum_{t=0}^r m^t \right) = K.$$

Since  $|u_{1+r}^{d-r}(m) - h_{1+r}^{d-r}(m)| < \epsilon_1$ , we have

$$\begin{aligned} & \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t v_r(t) \right) h_{1+r}^{d-r}(m) \\ (6.18) \quad & \approx \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) h_{1+r}^{d-r}(m) \\ & \approx \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) u_{1+r}^{d-r}(m) + \Delta(K\epsilon_1). \end{aligned}$$

Then we have

$$\begin{aligned} & \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) u_{1+r}^{d-r}(m) \\ & = \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) \left( w_{1+r}^{d-r}(m) - \sum_{r_1=1}^{d-r-1} w_{1+r}^{r_1}(m) h_{1+r+r_1}^{d-r-r_1}(m) \right). \end{aligned}$$

From (6.16) we have

$$\begin{aligned} & \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) w_{1+r}^{d-r}(m) \\ & = \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) \left( \frac{(mn)^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d + \sum_{j=0}^{d-r-1} m^j a_{1+r}^{d-r}(j) \right) \\ & = \sum_{t=0}^r \left( \sum_{m=1}^d \lambda_m m^{d-r+t} \right) \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d (v_r(t) - \lceil v_r(t) \rceil) \\ & \quad + \sum_{h=1}^{d-1} \left( \sum_{m=1}^d \lambda_m m^h \right) \left( \sum_{\substack{0 \leq t \leq r \\ 0 \leq j \leq d-r-1 \\ t+j=h}} (v_r(t) - \lceil v_r(t) \rceil) a_{1+r}^{d-r}(j) \right) \\ & \quad + \sum_{m=1}^d \lambda_m (v_r(0) - \lceil v_r(0) \rceil) a_{1+r}^{d-r}(0), \end{aligned}$$

and so  $\sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) w_{1+r}^{d-r}(m)$  is equal to

$$\begin{aligned} & \lambda \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d (v_r(r) - \lceil v_r(r) \rceil) + \left( \sum_{m=1}^d \lambda_m (v_r(0) - \lceil v_r(0) \rceil) \right) y_{1+r}^{d-r} \\ & \approx \lambda \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d (v_r(r) - \lceil v_r(r) \rceil) + \Delta (K \epsilon_1). \end{aligned}$$

The last equation follows from

$$\left| \left( \sum_{m=1}^d \lambda_m (v_r(0) - \lceil v_r(0) \rceil) \right) y_{1+r}^{d-r} \right| \leq \sum_{m=1}^d |\lambda_m| \epsilon_1 < K \epsilon_1.$$

Then for  $1 \leq r_1 \leq d - r - 1$  and  $m = 1, 2, \dots, d$ , by (6.16), we have

$$\begin{aligned} & \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) w_{1+r}^{r_1}(m) \\ & = \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) \left( \frac{(mn)^{r_1}}{(r_1)!} \alpha_{1+r} \dots \alpha_{r+r_1} + \sum_{j=0}^{r_1-1} m^j a_{1+r}^{r_1}(j) \right) \\ & = \sum_{t=0}^r m^{r_1+t} \frac{n^{r_1}}{(r_1)!} \alpha_{1+r} \dots \alpha_{r+r_1} (v_r(t) - \lceil v_r(t) \rceil) + \\ & \quad \sum_{h=0}^{r+r_1-1} m^h \left( \sum_{\substack{0 \leq t \leq r \\ 0 \leq j \leq r_1-1 \\ t+j=h}} (v_r(t) - \lceil v_r(t) \rceil) a_{1+r}^{r_1}(j) \right). \end{aligned}$$

Let

$$v_{r,r_1}(h) = \sum_{\substack{0 \leq t \leq r \\ 0 \leq j \leq r_1-1 \\ t+j=h}} (v_r(t) - \lceil v_r(t) \rceil) a_{1+r}^{r_1}(j)$$

for  $0 \leq h \leq r_1 - 1$ ,

$$\begin{aligned} v_{r,r_1}(h) & = \frac{n^{r_1}}{(r_1)!} \alpha_{1+r} \dots \alpha_{r+r_1} (v_r(h - r_1) - \lceil v_r(h - r_1) \rceil) + \\ & \quad \sum_{\substack{0 \leq t \leq r \\ 0 \leq j \leq r_1-1 \\ t+j=h}} (v_r(t) - \lceil v_r(t) \rceil) a_{1+r}^{r_1}(j) \end{aligned}$$

for  $r_1 \leq h \leq r + r_1 - 1$  and

$$v_{r,r_1}(r + r_1) = \frac{n^{r_1}}{(r_1)!} \alpha_{1+r} \dots \alpha_{r+r_1} (v_r(r) - \lceil v_r(r) \rceil).$$

Thus

$$\begin{aligned} & \sum_{r_1=1}^{d-r-1} \left( \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) w_{1+r}^{r_1}(m) h_{1+r+r_1}^{d-r-r_1}(m) \right) \\ &= \sum_{r_1=1}^{d-r-1} \left( \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^{r+r_1} m^t v_{r,r_1}(t) \right) h_{1+r+r_1}^{d-r-r_1}(m) \right). \end{aligned}$$

To sum up, we have

$$\begin{aligned} \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t (v_r(t) - \lceil v_r(t) \rceil) \right) u_{1+r}^{d-r}(m) &\approx \lambda \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d (v_r(r) - \lceil v_r(r) \rceil) \\ &\quad - \sum_{r_1=1}^{d-r-1} \left( \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^{r_1+r} m^t v_{r,r_1}(t) \right) h_{1+r+r_1}^{d-(r+r_1)}(m) \right) + \Delta(K\epsilon_1). \end{aligned}$$

Together with (6.18), we conclude

$$\begin{aligned} \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^r m^t v_r(t) \right) h_{1+r}^{d-r}(m) &\approx \lambda \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \dots \alpha_d (v_r(r) - \lceil v_r(r) \rceil) \\ &\quad - \sum_{r_1=1}^{d-r-1} \left( \sum_{m=1}^d \lambda_m \left( \sum_{t=0}^{r_1+r} m^t v_{r,r_1}(t) \right) h_{1+r+r_1}^{d-(r+r_1)}(m) \right) + \Delta(2K\epsilon_1). \end{aligned}$$

The proof of the claim is completed.  $\square$

We will use the claim repeatedly. First using (6.17) we have

$$\sum_{m=1}^d \lambda_m z_1^d(m) \approx \sum_{m=1}^d \lambda_m \left( w_1^d(m) - \sum_{j_1=1}^{d-1} w_1^{j_1}(m) h_{1+j_1}^{d-j_1}(m) \right).$$

By (6.16), we have

$$\sum_{m=1}^d \lambda_m w_1^d(m) = \sum_{m=1}^d \lambda_m m^d \frac{n^d}{d!} \alpha_1 \dots \alpha_d + \sum_{m=1}^d \lambda_m y_1^d \approx \lambda \frac{n^d}{d!} \alpha_1 \dots \alpha_d + \Delta(K\epsilon_1).$$

Using this, (6.16) and the claim, we have

$$\begin{aligned} \sum_{m=1}^d \lambda_m z_1^d(m) &\approx \lambda \frac{n^d}{d!} \alpha_1 \dots \alpha_d - \sum_{m=1}^d \lambda_m \sum_{j_1=1}^{d-1} \left( m^{j_1} \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} + \sum_{t=0}^{j_1-1} m^t a_1^{j_1}(t) \right) h_{1+j_1}^{d-j_1}(m) + \Delta(K\epsilon_1) \\ &\approx \lambda \frac{n^d}{d!} \alpha_1 \dots \alpha_d - \left( \sum_{j_1=1}^{d-1} \lambda \frac{n^{d-j_1}}{(d-j_1)!} \alpha_{1+j_1} \dots \alpha_d \left( \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} - \lceil \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} \rceil \right) \right) \\ &\quad + \sum_{j_1=1}^{d-1} \sum_{j_2=1}^{d-j_1-1} \left( \sum_{m=1}^d \lambda_m \left( m^{j_1+j_2} \left( \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} - \lceil \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} \rceil \right) \frac{n^{j_2}}{j_2!} \alpha_{1+j_1} \dots \alpha_{j_1+j_2} \right. \right. \\ &\quad \left. \left. + \sum_{t=0}^{j_1+j_2-1} m^t v_{j_1,j_2}(t) \right) h_{1+j_1+j_2}^{d-(j_1+j_2)}(m) \right) + \Delta((2(d-1)K + K)\epsilon_1). \end{aligned}$$

Note that here we use  $v_{j_1}(t) = a_1^{j_1}(t)$ ,  $t = 0, 1, \dots, j_1 - 1$  and  $v_{j_1}(j_1) = \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1}$ .

Recall the definition of  $U(\cdot)$ :

$$\frac{n^d}{d!} \alpha_1 \dots \alpha_d = U(n; d),$$

$$\left( \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} - \lceil \frac{n^{j_1}}{j_1!} \alpha_1 \dots \alpha_{j_1} \rceil \right) \frac{n^{j_2}}{j_2!} \alpha_{1+j_1} \dots \alpha_{j_1+j_2} = U(n; j_1, j_2).$$

Substituting these in the above equation, we have

$$\begin{aligned} \sum_{m=1}^d \lambda_m z_1^d(m) &\approx \lambda U(n; d) - \sum_{j_1=1}^{d-1} \lambda U(n; j_1, d-j_1) + \Delta(2dK\epsilon_1) \\ &\quad + \sum_{j_1=1}^{d-1} \sum_{j_2=1}^{d-j_1-1} \left( \sum_{m=1}^d \lambda_m \left( m^{j_1+j_2} U(n; j_1, j_2) + \sum_{t=0}^{j_1+j_2-1} m^t v_{j_1, j_2}(t) \right) h_{1+j_1+j_2}^{d-(j_1+j_2)}(m) \right) \end{aligned}$$

Using the claim again, we have:

$$\begin{aligned} \sum_{m=1}^d \lambda_m z_1^d(m) &\approx \lambda U(n; d) - \sum_{j_1=1}^{d-1} \lambda U(n; j_1, d-j_1) + \sum_{j_1=1}^{d-1} \sum_{j_2=1}^{d-j_1-1} \lambda U(n; j_1, j_2, d-j_1-j_2) \\ &\quad - \sum_{j_1=1}^{d-1} \sum_{j_2=1}^{d-j_1-1} \sum_{j_3=1}^{d-(j_1+j_2)-1} \left( \sum_{m=1}^d \lambda_m \left( m^{j_1+j_2+j_3} U(n; j_1, j_2, j_3) + \right. \right. \\ &\quad \left. \left. \sum_{t=0}^{j_1+j_2+j_3-1} m^t v_{j_1, j_2, j_3}(t) \right) h_{1+j_1+j_2+j_3}^{d-(j_1+j_2+j_3)}(m) \right) + \Delta(2dK\epsilon_1 + 2d^2K\epsilon_1). \end{aligned}$$

Inductively, we have

$$\begin{aligned} \sum_{m=1}^d \lambda_m z_1^d(m) &\approx \left( \sum_{l=1}^d (-1)^{l-1} \sum_{\substack{j_1, \dots, j_l \in \mathbb{N} \\ j_1 + \dots + j_l = d}} \lambda U(n; j_1, \dots, j_l) \right) + \Delta(2dK\epsilon_1 + 2d^2K\epsilon_1 + \dots + 2d^{d-1}K\epsilon_1) \\ &\approx \left( \sum_{l=1}^d (-1)^{l-1} \sum_{\substack{j_1, \dots, j_l \in \mathbb{N} \\ j_1 + \dots + j_l = d}} \lambda U(n; j_1, \dots, j_l) \right) + \Delta((d + d^2 + \dots + d^{d-1})(2K\epsilon_1)). \end{aligned}$$

The proof of (6.14) is now finished.  $\square$

## CHAPTER 7

### Recurrence sets and regionally proximal relation of order $d$

From this chapter we begin the study of higher order almost automorphy. In this chapter we investigate the relationship between recurrence sets and  $\mathbf{RP}^{[d]}$ . Then using the results developed in this chapter, one can characterize higher order almost automorphy in the next chapter.

#### 7.1. Regionally proximal relation of order $d$

**7.1.1. Cubes and faces.** In the following subsections, we will introduce notions about cubes, faces and face transformations. For more details see [34, 36].

7.1.1.1. Let  $X$  be a set, let  $d \geq 1$  be an integer, and write  $[d] = \{1, 2, \dots, d\}$ . We view  $\{0, 1\}^d$  in one of two ways, either as a sequence  $\epsilon = \epsilon_1 \dots \epsilon_d$  of 0's and 1's, or as a subset of  $[d]$ . A subset  $\epsilon$  corresponds to the sequence  $(\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$  such that  $i \in \epsilon$  if and only if  $\epsilon_i = 1$  for  $i \in [d]$ . For example,  $\mathbf{0} = (0, 0, \dots, 0) \in \{0, 1\}^d$  is the same as  $\emptyset \subset [d]$ .

If  $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$  and  $\epsilon \in \{0, 1\}^d$ , we define  $\mathbf{n} \cdot \epsilon = \sum_{i=1}^d n_i \epsilon_i$ . If we consider  $\epsilon$  as  $\epsilon \subset [d]$ , then  $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$ .

7.1.1.2. We denote  $X^{2^d}$  by  $X^{[d]}$ . A point  $\mathbf{x} \in X^{[d]}$  can be written in one of two equivalent ways, depending on the context:  $\mathbf{x} = (x_\epsilon : \epsilon \in \{0, 1\}^d) = (x_\epsilon : \epsilon \subset [d])$ . Hence  $x_\emptyset = x_{\mathbf{0}}$  is the first coordinate of  $\mathbf{x}$ . For example, points in  $X^{[2]}$  are like

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}).$$

For  $x \in X$ , we write  $x^{[d]} = (x, x, \dots, x) \in X^{[d]}$ . The diagonal of  $X^{[d]}$  is  $\Delta^{[d]} = \{x^{[d]} : x \in X\}$ . Usually, when  $d = 1$ , denote the diagonal by  $\Delta_X$ . A point  $\mathbf{x} \in X^{[d]}$  can be decomposed as  $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$  with  $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$ , where  $\mathbf{x}' = (x_{\epsilon 0} : \epsilon \in \{0, 1\}^{d-1})$  and  $\mathbf{x}'' = (x_{\epsilon 1} : \epsilon \in \{0, 1\}^{d-1})$ . We can also isolate the first coordinate, writing  $X_*^{[d]} = X^{2^{d-1}}$  and then writing a point  $\mathbf{x} \in X^{[d]}$  as  $\mathbf{x} = (x_\emptyset, \mathbf{x}_*)$ , where  $\mathbf{x}_* = (x_\epsilon : \epsilon \neq \emptyset) \in X_*^{[d]}$ .

#### 7.1.2. Face transformations.

**DEFINITION 7.1.1.** Let  $\phi : X \rightarrow Y$  and  $d \in \mathbb{N}$ . Define  $\phi^{[d]} : X^{[d]} \rightarrow Y^{[d]}$  by  $(\phi^{[d]} \mathbf{x})_\epsilon = \phi x_\epsilon$  for every  $\mathbf{x} \in X^{[d]}$  and every  $\epsilon \subset [d]$ . Let  $(X, T)$  be a system and  $d \geq 1$  be an integer. The *diagonal transformation* of  $X^{[d]}$  is the map  $T^{[d]}$ . *Face transformations* are defined inductively as follows: Let  $T^{[0]} = T$ ,  $T_1^{[1]} = \text{id} \times T$ . If  $\{T_j^{[d-1]}\}_{j=1}^{d-1}$  is defined already, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad j \in \{1, 2, \dots, d-1\},$$

$$T_d^{[d]} = \text{id}^{[d-1]} \times T^{[d-1]}.$$

The *face group* of dimension  $d$  is the group  $\mathcal{F}^{[d]}(X)$  of transformations of  $X^{[d]}$  generated by the face transformations. We often write  $\mathcal{F}^{[d]}$  instead of  $\mathcal{F}^{[d]}(X)$ . For  $\mathcal{F}^{[d]}$ , we use similar notations to that used for  $X^{[d]}$ : namely, an element of the group is written as  $S = (S_\epsilon : \epsilon \in \{0, 1\}^d)$ . For convenience, we denote the orbit closure of  $\mathbf{x} \in X^{[d]}$  under  $\mathcal{F}^{[d]}$  by  $\overline{\mathcal{F}^{[d]}(\mathbf{x})}$ , instead of  $\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})$ .

**7.1.3. Regionally proximal pairs of order  $d$ .** First let us define regionally proximal pairs of order  $d$ .

**DEFINITION 7.1.2.** Let  $(X, T)$  be a t.d.s. and let  $d \geq 1$  be an integer. A pair  $(x, y) \in X \times X$  is said to be *regionally proximal of order  $d$*  if for any  $\delta > 0$ , there exist  $x', y' \in X$  and a vector  $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$  such that  $\rho(x, x') < \delta, \rho(y, y') < \delta$ , and

$$\rho(T^{\mathbf{n} \cdot \epsilon} x', T^{\mathbf{n} \cdot \epsilon} y') < \delta \text{ for any nonempty } \epsilon \subset [d].$$

The set of regionally proximal pairs of order  $d$  is denoted by  $\mathbf{RP}^{[d]}$  (or by  $\mathbf{RP}^{[d]}(X)$  in case of ambiguity), which is called *the regionally proximal relation of order  $d$* .

Moreover, let  $\mathbf{RP}^{[\infty]} = \bigcap_{d=1}^{\infty} \mathbf{RP}^{[d]}(X)$ . The following theorem was proved by Host-Kra-Maass for minimal distal systems [36] and by Shao-Ye for general minimal systems, see Theorems 3.1, 3.5, 3.11, Proposition 6.1 and Theorem 6.4 in [48].

**THEOREM 7.1.3.** Let  $(X, T)$  be a minimal t.d.s. and  $d \in \mathbb{N}$ . Then

- (1)  $(x, y) \in \mathbf{RP}^{[d]}$  if and only if  $(x, y, y, \dots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$  if and only if  $(x, x_*^{[d]}, y, y_*^{[d]}) \in \overline{\mathcal{F}^{[d+1]}(x^{[d+1]})}$ .
- (2)  $(\overline{\mathcal{F}^{[d]}(x^{[d]})}, \mathcal{F}^{[d]})$  is minimal for all  $x \in X$ .
- (3)  $\mathbf{RP}^{[d]}(X)$  is an equivalence relation, and so is  $\mathbf{RP}^{[\infty]}$ .
- (4) If  $\pi : (X, T) \rightarrow (Y, S)$  is a factor map, then  $(\pi \times \pi)(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$ .
- (5)  $(X/\mathbf{RP}^{[d]}, T)$  is the maximal  $d$ -step nilfactor of  $(X, T)$ .

Note that (5) means that  $(X/\mathbf{RP}^{[d]}, T)$  is a system of order  $d$  and any system of order  $d$  factor of  $(X, T)$  is a factor of  $(X/\mathbf{RP}^{[d]}, T)$ .

**REMARK 7.1.4.** In [48], Theorem 7.1.3 was proved for compact metric spaces. In fact, one can show that Theorem 7.1.3 holds for compact Hausdorff spaces by repeating the proofs sentence by sentence in [48]. However, we will describe a direct approach in Appendix A.2. This result will be used in the next section.

## 7.2. Nil $_d$ Bohr $_0$ -sets, Poincaré sets and $\mathbf{RP}^{[d]}$

In this section using Theorem A we characterize  $\mathbf{RP}^{[d]}$  using the families  $\mathcal{F}_{\text{Poi}_d}$ ,  $\mathcal{F}_{\text{Bir}_d}$  and  $\mathcal{F}_{d,0}^*$ .

**7.2.1. Nil Bohr-sets.** Recall  $\mathcal{F}_{d,0}$  is the family consisting of all Nil<sub>d</sub> Bohr<sub>0</sub>-sets.

For  $F_1, F_2 \in \mathcal{F}_{d,0}$ , there are  $d$ -step nilsystems  $(X, T)$ ,  $(Y, S)$ ,  $(x, y) \in X \times Y$  and  $U \times V$  neighborhood of  $(x, y)$  such that  $N(x, U) \subset F_1$  and  $N(y, V) \subset F_2$ . It is clear that  $N(x, U) \cap N(y, V) = N((x, y), U \times V) \in \mathcal{F}_{d,0}$ . This implies that  $F_1 \cap F_2 \in \mathcal{F}_{d,0}$ . So we conclude that

**PROPOSITION 7.2.1.** Let  $d \in \mathbb{N}$ . Then  $\mathcal{F}_{d,0}$  is a filter, and  $\mathcal{F}_{d,0}^*$  has the Ramsey property.

**7.2.2. Sets of  $d$ -recurrence.**

7.2.2.1. Recall that for  $d \in \mathbb{N}$ ,  $\mathcal{F}_{\text{Poi}_d}$  (resp.  $\mathcal{F}_{\text{Bir}_d}$ ) is the family consisting of all sets of  $d$ -recurrence (resp. sets of  $d$ -topological recurrence).

**REMARK 7.2.2.** It is known that for all integer  $d \geq 2$  there exists a set of  $(d-1)$ -recurrence that is not a set of  $d$ -recurrence [18]. This also follows from Theorem 7.2.7.

Recall that a set  $S \subset \mathbb{Z}$  is  $d$ -intersective if every subset  $A$  of  $\mathbb{Z}$  with positive density contains at least one arithmetic progression of length  $d+1$  and a common difference in  $S$ , i.e. there is some  $n \in S$  such that

$$A \cap (A - n) \cap (A - 2n) \dots \cap (A - dn) \neq \emptyset.$$

Similarly, one can define topological  $d$ -intersective set by replacing the set with positive density by a syndetic set in the above definition.

We now give some equivalence conditions of  $d$ -topological recurrence.

**PROPOSITION 7.2.3.** Let  $S \subset \mathbb{Z}$ . Then the following statements are equivalent:

- (1)  $S$  is a set of topological  $d$ -intersective.
- (2)  $S$  is a set of  $d$ -topological recurrence.
- (3) For any t.d.s.  $(X, T)$  there are  $x \in X$  and  $\{n_i\}_{i=1}^\infty \subset S$  such that

$$\lim_{i \rightarrow +\infty} T^{jn_i} x = x \text{ for each } 1 \leq j \leq d.$$

**PROOF.** The equivalence between (1) and (2) was proved in [18, 20].

(2)  $\Rightarrow$  (3). Now assume that whenever  $(Y, S)$  is a minimal t.d.s. and  $V \subset Y$  a nonempty open set, there is  $n \in S$  such that

$$V \cap T^{-n}V \cap \dots \cap T^{-dn}V \neq \emptyset.$$

Let  $(X, T)$  be a t.d.s., and without loss of generality we assume that  $(X, T)$  is minimal, since each t.d.s. contains a minimal subsystem. Define for each  $j \in \mathbb{N}$

$$W_j = \{x \in X : \exists n \in S \text{ with } d(T^{kn}x, x) < \frac{1}{j} \text{ for each } 1 \leq k \leq d\}.$$

Then it is easy to verify that  $W_j$  is non-empty, open and dense. Then any  $x \in \bigcap_{j=1}^\infty W_j$  is the point we look for.

(3)  $\Rightarrow$  (2). Let  $(X, T)$  be a minimal t.d.s. and  $U \subset X$  a nonempty open set. Then there are  $x \in X$  and  $\{n_i\}_{i=1}^\infty \subset S$  such that for each given  $1 \leq k \leq d$ ,  $T^{kn_i}x \rightarrow x$ . Since  $(X, T)$  is minimal, there is some  $l \in \mathbb{Z}$  such that  $x \in V = T^{-l}U$ . When

$i_0$  is larger enough, we have  $V \cap T^{-n_{i_0}}V \cap \dots \cap T^{-dn_{i_0}}V \neq \emptyset$ , which implies that  $U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset$  by putting  $n = n_{i_0}$ .  $\square$

7.2.2.2. The following fact follows from the Poincaré and Birkhoff multiple recurrent theorems.

PROPOSITION 7.2.4. For all  $d \in \mathbb{N}$ ,  $\mathcal{F}_{Poi_d}$  and  $\mathcal{F}_{Bir_d}$  have the Ramsey property.

PROOF. Let  $F \in \mathcal{F}_{Poi_d}$  and  $F = F_1 \cup F_2$ . Assume the contrary that  $F_i \notin \mathcal{F}_{Poi_d}$  for  $i = 1, 2$ . Then there are measure preserving systems  $(X_i, \mathcal{B}_i, \mu_i, T_i)$  and  $A_i \in \mathcal{B}_i$  with  $\mu_i(A_i) > 0$  such that  $\mu_i(A_i \cap T_i^{-n}A_i \cap \dots \cap T_i^{-dn}A_i) = 0$  for  $n \in F_i$ , where  $i = 1, 2$ . Set  $X = X_1 \times X_2$ ,  $\mu = \mu_1 \times \mu_2$ ,  $A = A_1 \times A_2$  and  $T = T_1 \times T_2$ . Then we have

$$\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) = \mu_1\left(\bigcap_{i=0}^d T_1^{-in}A_1\right)\mu_2\left(\bigcap_{i=0}^d T_2^{-in}A_2\right) = 0$$

for each  $n \in F = F_1 \cup F_2$ , a contradiction.

Now let  $F \in \mathcal{F}_{Bir_d}$  and  $F = F_1 \cup F_2$ . Assume the contrary that  $F_i \notin \mathcal{F}_{Bir_d}$  for  $i = 1, 2$ . Then there are minimal systems  $(X_i, T_i)$  and non-empty open subsets  $U_i$  such that  $U_i \cap T_i^{-n}U_i \cap \dots \cap T_i^{-dn}U_i = \emptyset$  for  $n \in F_i$ , where  $i = 1, 2$ . Let  $X$  be a minimal subset of  $X_1 \times X_2$ ,  $U = (U_1 \times U_2) \cap X$  and  $T = T_1 \times T_2$ . Replacing  $U_1$  and  $U_2$  by  $T_1^{-i_1}U_1$  and  $T_2^{-i_2}U_2$  respectively (if necessary) we may assume that  $U \neq \emptyset$  (using the minimality of  $T_1$  and  $T_2$ ). Then we have

$$U \cap T^{-n}U \cap \dots \cap T^{-dn}U \subset \bigcap_{i=0}^d T_1^{-in}U_1 \times \bigcap_{i=0}^d T_2^{-in}U_2 = \emptyset$$

for each  $n \in F = F_1 \cup F_2$ , a contradiction.  $\square$

**7.2.3. Nil<sub>d</sub> Bohr<sub>0</sub>-sets and  $\mathbf{RP}^{[d]}$ .** To show the following result we need several well known facts (related to distality) from the Ellis enveloping semigroup theory, see [2, 53]. Also we note that the lifting property in Theorem 7.1.3 is valid when  $X$  is compact and Hausdorff (see Appendix A.2 for more details).

THEOREM 7.2.5. Let  $(X, T)$  be a minimal t.d.s. Then  $(x, y) \in \mathbf{RP}^{[d]}$  if and only if  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ .

PROOF. First assume that  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ . Let  $(X_d, S)$  be the maximal  $d$ -step nilfactor of  $(X, T)$  (see Theorem 7.1.3) and  $\pi : X \rightarrow X_d$  be the projection. Then for any neighborhood  $V$  of  $\pi(x)$ , we have  $N(x, U) \cap N(\pi(x), V) \neq \emptyset$  since  $N(x, U) \in \mathcal{F}_{d,0}^*$ . This means that there is a sequence  $\{n_i\}$  such that

$$(T \times S)^{n_i}(x, \pi(x)) \rightarrow (y, \pi(x)), \quad i \rightarrow \infty.$$

Thus, we have

$$\pi(y) = \pi(\lim_i T^{n_i}x) = \lim_i S^{n_i}\pi(x) = \pi(x),$$

i.e.  $(x, y) \in \mathbf{RP}^{[d]}$ .



Now assume that  $(x, y) \in \mathbf{RP}^{[d]}$  and  $U$  is a neighborhood of  $y$ . We need to show that if  $(Z, R)$  is a  $d$ -step nilsystem,  $z_0 \in Z$  and  $V$  is a neighborhood of  $z_0$  then  $N(x, U) \cap N(z_0, V) \neq \emptyset$ .

Let

$$W = \prod_{z \in Z} Z \quad (\text{i.e. } W = Z^Z) \text{ and } R^Z : W \rightarrow W$$

with  $(R^Z \omega)(z) = R(\omega(z))$  for any  $z \in Z$ , where  $\omega = (\omega(z))_{z \in Z} \in W$ . Note that in general  $(W, R^Z)$  is not a metrizable but a compact Hausdorff system. Since  $(Z, R)$  is a  $d$ -step nilsystem,  $(Z, R)$  is distal. Hence  $(W, R^Z)$  is also distal.

Choose  $\omega^* \in W$  with  $\omega^*(z) = z$  for all  $z \in Z$ , and let  $Z_\infty = \overline{\mathcal{O}(\omega^*, R^Z)}$ . Then  $(Z_\infty, R^Z)$  is a minimal subsystem of  $(W, R^Z)$  since  $(W, R^Z)$  is distal. For any  $\omega \in Z_\infty$ , there exists  $p \in E(Z, R)$  such that  $\omega(z) = p(\omega^*(z)) = p(z)$  for all  $z \in Z$ . Since  $(Z, R)$  is a distal system, the Ellis semigroup  $E(Z, R)$  is a group (Appendix A.2). Particularly,  $p : Z \rightarrow Z$  is a surjective map. Thus

$$\{\omega(z) : z \in Z\} = \{p(z) : z \in Z\} = Z.$$

Hence there exists  $z_\omega \in Z$  such that  $\omega(z_\omega) = z_0$ .

Take a minimal subsystem  $(A, T \times R^Z)$  of the product system  $(X \times Z_\infty, T \times R^Z)$ . Let  $\pi_X : A \rightarrow X$  be the natural coordinate projection. Then  $\pi_X : (A, T \times R^Z) \rightarrow (X, T)$  is a factor map between two minimal systems. Since  $(x, y) \in \mathbf{RP}^{[d]}(X, T)$ , by Theorem 7.1.3 there exist  $\omega^1, \omega^2 \in W$  such that  $((x, \omega^1), (y, \omega^2)) \in \mathbf{RP}^{[d]}(A, T \times R^Z)$ .

For  $\omega^1$ , there exists  $z_1 \in Z$  such that  $\omega^1(z_1) = z_0$  by the above discussion. Let  $\pi : A \rightarrow X \times Z$  with  $\pi(u, \omega) = (u, \omega(z_1))$  for  $(u, \omega) \in A$ ,  $u \in X$ ,  $\omega \in W$ . Let  $B = \pi(A)$ . Then  $(B, T \times R)$  is a minimal subsystem of  $(X \times Z, T \times R)$ , and  $\pi : (A, T \times R^Z) \rightarrow (B, T \times R)$  is a factor map between two minimal systems. Clearly  $\pi(x, \omega^1) = (x, z_0)$ ,  $\pi(y, \omega^2) = (y, z_2)$  for some  $z_2 \in Z$ , and

$$((x, z_0), (y, z_2)) = \pi \times \pi((x, \omega^1), (y, \omega^2)) \in \mathbf{RP}^{[d]}(B, T \times R).$$

Moreover, we consider the projection  $\pi_Z$  of  $B$  onto  $Z$ . Then  $\pi_Z : (B, T \times R) \rightarrow (Z, R)$  is a factor map and so  $(z_0, z_2) = \pi_Z \times \pi_Z((x, z_0), (y, z_2)) \in \mathbf{RP}^{[d]}(Z, R)$ . Since  $(Z, R)$  is a system of order  $d$ ,  $z_0 = z_2$ . Thus  $((x, z_0), (y, z_0)) \in B$ . Particularly,  $N(x, U) \cap N(z_0, V) = N((x, z_0), U \times V)$  is a syndetic set since  $(B, T \times R)$  is minimal. This completes the proof of theorem.  $\square$

**REMARK 7.2.6.** From the proof of Theorem 7.2.5, we have the following result: if  $(X, T)$  is minimal and  $(x, y) \in \mathbf{RP}^{[d]}$  then  $N(x, U) \cap F$  is a syndetic set for each  $F \in \mathcal{F}_{d,0}$  and each neighborhood  $U$  of  $y$ .

**7.2.4. Recurrence sets and  $\mathbf{RP}^{[d]}$ .** Now we can sum up the main result of this section as follows, whose proof depends on Theorem A.

**THEOREM 7.2.7.** Let  $(X, T)$  be a minimal t.d.s.,  $d \in \mathbb{N}$  and  $x, y \in X$ . Then the following statements are equivalent:

- (1)  $(x, y) \in \mathbf{RP}^{[d]}$ .
- (2)  $N(x, U) \in \mathcal{F}_{Poi_d}$  for each neighborhood  $U$  of  $y$ .
- (3)  $N(x, U) \in \mathcal{F}_{Bir_d}$  for each neighborhood  $U$  of  $y$ .

(4)  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ .

PROOF. First we show that (1)  $\Rightarrow$  (2). Let  $U$  be a neighborhood of  $y$ . We need to show  $N(x, U) \in \mathcal{F}_{Poi_d}$ .

Now let  $(Y, \mathcal{Y}, \mu, S)$  be a measure preserving system and  $A \in \mathcal{Y}$  with  $\mu(A) > 0$ . Let  $\mu = \int_{\Omega} \mu_{\omega} dm(\omega)$  be an ergodic decomposition of  $\mu$ . Then there is  $\Omega' \subset \Omega$  with  $m(\Omega') > 0$  such that for each  $\omega \in \Omega'$ ,  $\mu_{\omega}(A) > 0$ . For  $\omega \in \Omega'$ , set

$$F_{\omega} = \{n \in \mathbb{Z} : \mu_{\omega}(A \cap S^{-n}A \cap \dots \cap S^{-dn}A) > 0\}.$$

By Theorem 2.2.4 there is some subset  $M$  with  $BD^*(M) = 0$  such that  $B = F_{\omega} \Delta M$  is a  $\text{Nil}_d$  Bohr $_0$ -set. Hence we have  $N(x, U) \cap (F_{\omega} \Delta M)$  is syndetic by Remark 7.2.6. Thus we conclude that there is  $n_{\omega} \neq 0$  with  $n_{\omega} \in N(x, U) \cap F_{\omega}$  since  $BD^*(M) = 0$ . This implies that there are  $\Omega'' \subset \Omega'$  with  $m(\Omega'') > 0$  and  $n \in N(x, U)$  such that for each  $\omega \in \Omega''$  one has  $\mu_{\omega}(A \cap S^{-n}A \cap \dots \cap S^{-dn}A) > 0$  which in turn implies  $\mu(A \cap S^{-n}A \cap \dots \cap S^{-dn}A) > 0$ . By the definition,  $N(x, U) \in \mathcal{F}_{Poi_d}$ .

It follows from Corollary D that (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4). By Theorem 7.2.5, one has that (4)  $\Rightarrow$  (1) and completes the proof.  $\square$

### 7.3. $SG_d$ -sets and $\mathbf{RP}^{[d]}$

In this section we will describe  $\mathbf{RP}^{[d]}$  using the  $SG_d$ -sets introduced by Host and Kra in [35]. First we recall some definitions.

**7.3.1. Sets  $SG_d(P)$ .** Recall that for  $d \in \mathbb{N}$  and a (finite or infinite) sequence  $P = \{p_i\}_i \in \mathbb{Z}$  the *set of sums with gaps of length less than  $d$*  of  $P$  is the set  $SG_d(P)$  of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \dots + \epsilon_n p_n$$

where  $n \geq 1$  is an integer,  $\epsilon_i \in \{0, 1\}$  for  $1 \leq i \leq n$ , the  $\epsilon_i$  are not all equal to 0, and the blocks of consecutive 0's between two 1 have length less than  $d$ .

Note that in this definition,  $P$  is a sequence and not a subset of  $\mathbb{Z}$ . For example, if  $P = \{p_i\}$ , then  $SG_1(P)$  is the set of all sums  $p_m + p_{m+1} + \dots + p_n$  of consecutive elements of  $P$ , and thus it coincides with the set  $\Delta(S)$  where  $S = \{0, p_1, p_1 + p_2, p_1 + p_2 + p_3, \dots\}$ . Therefore  $SG_1^*$ -sets are the same as  $\Delta^*$ -sets.

For a sequence  $P$ ,  $SG_2(P)$  consists of all sums of the form

$$\sum_{i=m_0}^{m_1} p_i + \sum_{i=m_1+2}^{m_2} p_i + \dots + \sum_{i=m_{k-1}+2}^{m_k} p_i + \sum_{i=m_k+2}^{m_{k+1}} p_i$$

where  $k \in \mathbb{N}$  and  $m_0, m_1, \dots, m_{k+1}$  are positive integers satisfying  $m_{i+1} \geq m_i + 2$  for  $i = 1, \dots, k$ , and  $m_1 \geq m_0$ .

Recall that for each  $d \in \mathbb{N}$ ,  $\mathcal{F}_{SG_d}$  is the family generated by  $SG_d$ . Moreover, let  $\mathcal{F}_{fSG_d}$  be the family of sets containing arbitrarily long  $SG_d(P)$  sets with  $P$  finite. That is,  $A \in \mathcal{F}_{fSG_d}$  if and only if there are finite sequences  $P^i$  with  $|P^i| \rightarrow \infty$  such

that  $\bigcup_{i=1}^{\infty} SG_d(P^i) \subset A$ . It is clear that

$$\mathcal{F}_{SG_1} \supset \mathcal{F}_{SG_2} \supset \dots \supset \mathcal{F}_{SG_{\infty}} =: \bigcap_{i=1}^{\infty} \mathcal{F}_{SG_i},$$

and

$$\mathcal{F}_{fSG_1} \supset \mathcal{F}_{fSG_2} \supset \dots \supset \mathcal{F}_{fSG_{\infty}} =: \bigcap_{i=1}^{\infty} \mathcal{F}_{fSG_i}.$$

We now show

**PROPOSITION 7.3.1.** The following statements hold:

- (1)  $\mathcal{F}_{SG_{\infty}} = \{A : \exists P^i \text{ infinite for each } i \in \mathbb{N} \text{ such that } A \supset \bigcup_{i=1}^{\infty} SG_i(P^i)\}.$
- (2)  $\mathcal{F}_{fSG_{\infty}} = \mathcal{F}_{fip}.$

**PROOF.** (1). Assume that  $A \in \mathcal{F}_{SG_{\infty}}$ . Then  $A \in \bigcap_{i=1}^{\infty} \mathcal{F}_{SG_i}$  and hence  $A \in \mathcal{F}_{SG_i}$  for each  $i \in \mathbb{N}$ . Thus for each  $i \in \mathbb{N}$  there is  $P^i$  infinite such that  $A \supset SG_i(P^i)$  which implies that  $A \supset \bigcup_{i=1}^{\infty} SG_i(P^i)$ .

Now let  $B = \bigcup_{i=1}^{\infty} SG_i(P^i)$ , where  $P^i$  infinite for each  $i \in \mathbb{N}$ . It is clear that  $B \subset \mathcal{F}_{SG_i}$  for each  $i$  and thus,  $B \in \mathcal{F}_{SG_{\infty}}$ . Since  $\mathcal{F}_{SG_{\infty}}$  is a family, we conclude that  $\{A : \exists P^i \text{ infinite for each } i \in \mathbb{N} \text{ such that } A \supset \bigcup_{i=1}^{\infty} SG_i(P^i)\} \subset \mathcal{F}_{SG_{\infty}}$ .

(2) It is clear that  $\mathcal{F}_{fSG_{\infty}} \subset \mathcal{F}_{fip}$ . Let  $A \in \mathcal{F}_{fip}$  and without loss of generality assume that  $A = \bigcup_{i=1}^{\infty} FS(P^i)$  with  $P^i = \{p_1^i, \dots, p_i^i\}$  and  $|P^i| \rightarrow \infty$ .

Put  $A_d = \bigcup_{i=1}^{\infty} SG_d(P^i) \subset A$  for  $d \in \mathbb{N}$ . Then  $A_d \in \mathcal{F}_{fSG_d}$  which implies that  $A \in \mathcal{F}_{fSG_d}$  for each  $d \geq 1$  and hence  $A \in \mathcal{F}_{fSG_{\infty}}$ . That is,  $\mathcal{F}_{fip} \subset \mathcal{F}_{fSG_{\infty}}$ .  $\square$

**7.3.2.  $SG_d$ -sets and  $\mathbf{RP}^{[d]}$ .** The following theorem is the main result of this section.

**THEOREM 7.3.2.** Let  $(X, T)$  be a minimal t.d.s. Then for any  $d \in \mathbb{N}$ ,  $(x, y) \in \mathbf{RP}^{[d]}$  if and only if  $N(x, U) \in \mathcal{F}_{SG_d}$  for each neighborhood  $U$  of  $y$ . The same holds when  $d = \infty$ .

**PROOF.** It is clear that if  $N(x, U) \in \mathcal{F}_{SG_d}$  for each neighborhood  $U$  of  $y$ , then it contains some  $FS(\{n_i\}_{i=1}^{d+1})$  for each neighborhood  $U$  of  $y$  which implies that  $(x, y) \in \mathbf{RP}^{[d]}$  by Theorem 7.1.3.

Now assume that  $(x, y) \in \mathbf{RP}^{[d]}$  for  $d \geq 1$ . Let for  $i \geq 2$

$$A_i =: \{0, 1\}^i \setminus \{(0, \dots, 0, 0), (0, \dots, 0, 1)\}$$

The case when  $d = 1$  was proved by Veech [52] and our method is also valid for this case. To make the idea of the proof clearer, we first show the case when  $d = 2$  and the general case follows by the same idea.

#### I. The case $d = 2$ .

Assume that  $(x, y) \in \mathbf{RP}^{[2]}$ . Then by Theorem 7.1.3 (1) for each neighborhood  $V \times U$  of  $(x, y)$ , there are  $n_1, n_2, n_3 \in \mathbb{Z}$  such that

$$T^{\epsilon_1 n_1 + \epsilon_2 n_2 + \epsilon_3 n_3} x \in V \text{ and } T^{n_3} x \in U,$$

for each  $(\epsilon_1, \epsilon_2, \epsilon_3) \in A_3$ . For a given  $U$ , let  $\eta > 0$  with  $B(y, \eta) \subset U$ , and take  $\eta_i > 0$  with  $\sum_{i=1}^{\infty} \eta_i < \eta$ , where  $B(y, \eta) = \{z \in X : \rho(z, y) < \eta\}$ .

Choose  $n_1^1, n_2^1, n_3^1 \in \mathbb{Z}$  such that

$$\rho(T^{n_3^1}x, y) < \eta_1 \text{ and } \rho(T^r x, x) < \eta_1,$$

for each  $r \in E_1$  with

$$E_1 = \{\epsilon_1 n_1^1 + \epsilon_2 n_2^1 + \epsilon_3 n_3^1 : (\epsilon_1, \epsilon_2, \epsilon_3) \in A_3\}.$$

Let

$$S_1 = FS(\{n_1^1, n_2^1, n_3^1\}).$$

Choose  $n_1^2, n_2^2, n_3^2 \in \mathbb{Z}$  such that

$$\rho(T^{n_3^2}x, y) < \eta_2 \text{ and } \max_{s \in S_1} \rho(T^{s+r}x, T^s x) < \eta_2$$

for each  $r \in E_2$  with

$$E_2 = \{\epsilon_1 n_1^2 + \epsilon_2 n_2^2 + \epsilon_3 n_3^2 : (\epsilon_1, \epsilon_2, \epsilon_3) \in A_3\}.$$

Let

$$S_2 = FS(\{n_i^j : j = 1, 2, i = 1, 2, 3\}).$$

Generally when  $n_1^i, n_2^i, n_3^i, E_i, S_i$  are defined for  $1 \leq i \leq k$  choose  $n_1^{k+1}, n_2^{k+1}, n_3^{k+1} \in \mathbb{Z}$  such that

$$(7.1) \quad \rho(T^{n_3^{k+1}}x, y) < \eta_{k+1} \text{ and } \max_{s \in S_k} \rho(T^{s+r}x, T^s x) < \eta_{k+1}.$$

for each  $r \in E_{k+1}$ , where

$$E_{k+1} = \{\epsilon_1 n_1^{k+1} + \epsilon_2 n_2^{k+1} + \epsilon_3 n_3^{k+1} : (\epsilon_1, \epsilon_2, \epsilon_3) \in A_3\}.$$

Let

$$S_{k+1} = FS(\{n_i^j : i = 1, 2, 3, 1 \leq j \leq k+1\}).$$

Now we define a sequence  $P = \{P_k\}$  such that

$$P_1 = n_3^1 + n_1^2 + n_1^3, P_2 = n_3^2 + n_2^3 + n_2^4, P_3 = n_3^3 + n_1^4 + n_1^5, P_4 = n_3^4 + n_2^5 + n_2^6, \dots$$

That is,

$$P_k = n_3^k + n_{k \pmod{2}}^{k+1} + n_{k \pmod{2}}^{k+2},$$

where we set  $2m \pmod{2} = 2$  for  $m \in \mathbb{N}$ . We claim that  $N(x, U) \supset SG_2(P)$ .

Let  $n \in SG_2(P)$ . Then  $n = \sum_{j=1}^k P_{i_j}$ , where  $1 \leq i_{j+1} - i_j \leq 2$  for  $1 \leq j \leq k-1$ . By induction for  $k$ , it is not hard to show that  $n$  can be written as

$$n = a_1 + a_2 + \dots + a_{i_k - i_1 + 3}$$

such that  $a_1 = n_3^{i_1}$ ,  $a_j \in E_{j+i_1-1}$  for  $j = 2, 3, \dots, i_k - i_1 + 1$  and  $a_{i_k - i_1 + 2} \in \{n_1^{i_k+1}, n_2^{i_k+1}, n_1^{i_k+1} + n_2^{i_k+1}\}$ ,  $a_{i_k - i_1 + 3} = n_{i_k \pmod{2}}^{i_k+2}$ . In other words,  $n$  can be written as  $n = a_1 + a_2 + \dots + a_{i_k - i_1 + 3}$  with  $a_1 = n_3^{i_1}$  and  $a_j \in E_{i_1+j-1}$  for  $2 \leq j \leq i_k - i_1 + 3$ .

Note that  $\sum_{\ell=1}^j a_\ell \in S_{i_1+j-1}$  and  $a_{j+1} \in E_{i_1+j}$  for  $1 \leq j \leq i_k - i_1 + 2$ . Thus by (7.1) we have

$$\rho(T^{\sum_{i=1}^j a_i}x, T^{\sum_{i=1}^{j+1} a_i}x) < \eta_{j+i_1}$$

for  $1 \leq j \leq i_k - i_1 + 2$ . This implies that

$$\begin{aligned} \rho(T^n x, y) &\leq \rho(T^{\sum_{j=1}^{i_k - i_1 + 3} a_j} x, T^{\sum_{j=1}^{i_k - i_1 + 2} a_j} x) + \dots + \rho(T^{n_3^{i_1} + a_2} x, T^{n_3^{i_1}} x) + \rho(T^{n_3^{i_1}} x, y) \\ &< \sum_{j=0}^{i_k - i_1 + 2} \eta_{j+i_1} < \eta. \end{aligned}$$

That is,  $n \in N(x, U)$  and hence  $N(x, U) \supset SG_2(P)$ .

## II. The general case.

Generally assume that  $(x, y) \in \mathbf{RP}^{[d]}$  with  $d \geq 2$ . Then by Theorem 7.1.3 (1) for each neighborhood  $V \times U$  of  $(x, y)$ , there are  $n_1, n_2, \dots, n_{d+1} \in \mathbb{Z}$  such that

$$T^{\epsilon_1 n_1 + \epsilon_2 n_2 + \dots + \epsilon_{d+1} n_{d+1}} x \in V \text{ and } T^{n_{d+1}} x \in U,$$

for each  $(\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}$ . For a given  $U$ , let  $\eta > 0$  with  $B(y, \eta) \subset U$ , and take  $\eta_i > 0$  with  $\sum_{i=1}^{\infty} \eta_i < \eta$ .

Choose  $n_1^1, n_2^1, \dots, n_{d+1}^1 \in \mathbb{Z}$  such that  $\rho(T^{n_{d+1}^1} x, y) < \eta_1$  and  $\rho(T^r x, x) < \eta_1$  where  $r \in E_1$  with

$$E_1 = \{\epsilon_1 n_1^1 + \epsilon_2 n_2^1 + \dots + \epsilon_{d+1} n_{d+1}^1 : (\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}\}.$$

Let

$$S_1 = FS(\{n_1^1, \dots, n_{d+1}^1\}).$$

Choose  $n_1^2, n_2^2, \dots, n_{d+1}^2 \in \mathbb{Z}$  such that

$$\rho(T^{n_{d+1}^2} x, y) < \eta_2 \text{ and } \max_{s \in S_1} \rho(T^{s+r} x, T^s x) < \eta_2$$

for each  $r \in E_2$  with

$$E_2 = \{\epsilon_1 n_1^2 + \epsilon_2 n_2^2 + \dots + \epsilon_{d+1} n_{d+1}^2 : (\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}\}.$$

Let

$$S_2 = FS(\{n_1^1, \dots, n_{d+1}^1, n_1^2, \dots, n_{d+1}^2\}).$$

Generally when  $n_1^i, \dots, n_{d+1}^i$ ,  $E_i, S_i$  are defined for  $1 \leq i \leq k$  choose  $n_1^{k+1}, \dots, n_{d+1}^{k+1} \in \mathbb{Z}$  such that

$$(7.2) \quad \rho(T^{n_{d+1}^{k+1}} x, y) < \eta_{k+1} \text{ and } \max_{s \in S_k} \rho(T^{s+r} x, T^s x) < \eta_{k+1}.$$

for each  $r \in E_{k+1}$ , where

$$E_{k+1} = \{\epsilon_1 n_1^{k+1} + \epsilon_2 n_2^{k+1} + \dots + \epsilon_{d+1} n_{d+1}^{k+1} : (\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}\}.$$

Let

$$S_{k+1} = FS(\{n_i^j : i = 1, \dots, d+1, 1 \leq j \leq k+1\}).$$

Now we define a sequence  $P = \{P_k\}$  such that

$$\begin{aligned} P_1 &= n_{d+1}^1 + n_1^2 + \dots + n_1^{d+1}, P_2 = n_{d+1}^2 + n_2^3 + \dots + n_2^{d+2}, \dots, \\ P_d &= n_{d+1}^d + n_d^{d+1} + \dots + n_d^{2d}, \\ P_{d+1} &= n_{d+1}^{d+1} + n_1^{d+2} + \dots + n_1^{2d+1}, P_{d+2} = n_{d+1}^{d+2} + n_2^{d+3} + \dots + n_2^{2d+2}, \dots, \\ P_{2d} &= n_{d+1}^{2d} + n_d^{2d+1} + \dots + n_d^{3d}, \dots \end{aligned}$$

That is,

$$P_k = n_{d+1}^k + n_{k \pmod{d}}^{k+1} + \dots + n_{k \pmod{d}}^{k+d},$$

where we set  $dm \pmod{d} = d$  for  $m \in \mathbb{N}$ .

We claim that  $N(x, U) \supset SG_d(P)$ . Let  $n \in SG_d(P)$  then  $n = \sum_{j=1}^k P_{i_j}$ , where  $1 \leq i_{j+1} - i_j \leq d$  for  $1 \leq j \leq k-1$ . By induction for  $k$ , it is not hard to show that  $n$  can be written as

$$n = a_1 + a_2 + \dots + a_{i_k - i_1 + d + 1}$$

such that  $a_1 = n_{d+1}^{i_1}$ ,  $a_j \in E_{j+i_1-1}$  for  $j = 2, 3, \dots, i_k - i_1 + 1$  and

$$a_{i_k - i_1 + 1 + r} \in FS(\{n_\ell^{i_k + r} : \ell \in \{1, 2, \dots, d\} \setminus \cup_{j=1}^{r-1} \{i_k + j \pmod{d}\}\})$$

for  $1 \leq r \leq d$ . In other words,  $n$  can be written as  $n = a_1 + a_2 + \dots + a_{i_k - i_1 + d + 1}$  with  $a_1 = n_{d+1}^{i_1}$  and  $a_j \in E_{i_1+j-1}$  for  $2 \leq j \leq i_k - i_1 + d + 1$ .

Note that  $\sum_{\ell=1}^j a_\ell \in S_{i_1+j-1}$  and  $a_{j+1} \in E_{i_1+j}$  for  $1 \leq j \leq i_k - i_1 + d$ . Thus by (7.2) we have

$$\rho(T^{\sum_{i=1}^j a_i} x, T^{\sum_{i=1}^{j+1} a_i} x) < \eta_{i_1+j}$$

for  $1 \leq j \leq i_k - i_1 + d$ . This implies that

$$\begin{aligned} \rho(T^n x, y) &\leq \rho(T^{\sum_{j=1}^{i_k - i_1 + d + 1} a_j} x, T^{\sum_{j=1}^{i_k - i_1 + d} a_j} x) + \dots + \rho(T^{a_1} x, y) \\ &< \sum_{j=0}^{i_k - i_1 + d} \eta_{j+i_1} < \eta. \end{aligned}$$

That is,  $n \in N(x, U)$  and hence  $N(x, U) \supset SG_d(P)$  which implies that  $N(x, U) \in \mathcal{F}_{SG_d}$ . The proof is completed.  $\square$

#### 7.4. Cubic version of multiple recurrence sets and $\mathbf{RP}^{[d]}$

Cubic version of multiple ergodic averages was studied in [34], and also was proved very useful in some other questions [35, 36].

In this section we will discuss the question how to describe  $\mathbf{RP}^{[d]}$  using cubic version of multiple recurrence sets. Since by Theorem 7.1.3 one can use dynamical parallelepipeds to characterize  $\mathbf{RP}^{[d]}$ , it seems natural to describe  $\mathbf{RP}^{[d]}$  using the cubic version of multiple recurrence sets.

**7.4.1. Cubic version of multiple Birkhoff recurrence sets.** First we give definitions for the cubic version of multiple recurrence sets. We leave the equivalent statements in viewpoint of intersective sets to Appendix A.3.

**7.4.1.1. Birkhoff recurrence sets.** First we recall the classical definition. Let  $P \subset \mathbb{Z}$ .  $P$  is called a *Birkhoff recurrence set* (or a *set of topological recurrence*) if whenever  $(X, T)$  is a minimal t.d.s. and  $U \subset X$  a nonempty open set, then  $P \cap N(U, U) \neq \emptyset$ . Let  $\mathcal{F}_{Bir}$  denote the collection of Birkhoff recurrence subsets of  $\mathbb{Z}$ . An alternative definition is that for any t.d.s.  $(X, T)$  there are  $\{n_i\} \subset P$  and  $x \in X$  such that  $T^{n_i} x \rightarrow x$ . Now we generalize the above definition to the higher order.

**DEFINITION 7.4.1.** Let  $d \in \mathbb{N}$ . A subset  $P$  of  $\mathbb{Z}$  is called a *Birkhoff recurrence set of order  $d$*  (or a *set of topological recurrence of order  $d$* ) if whenever  $(X, T)$  is a t.d.s. there are  $x \in X$  and  $\{n_i^j\}_{j=1}^d \subset P$ ,  $i \in \mathbb{N}$ , such that  $FS(\{n_i^j\}_{j=1}^d) \subset P$ ,  $i \in \mathbb{N}$  and for each given  $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$ ,  $T^{m_i}x \rightarrow x$ , where  $m_i = \epsilon_1 n_i^1 + \dots + \epsilon_d n_i^d$ ,  $i \in \mathbb{N}$ . A subset  $F$  of  $\mathbb{Z}$  is a *Birkhoff recurrence set of order  $\infty$*  if it is a Birkhoff recurrence set of order  $d$  for any  $d \geq 1$ .

For example, when  $d = 2$  this means that there are sequence  $\{n_i\}, \{m_i\} \subset P$  and  $x \in X$  such that  $\{n_i + m_i\} \subset P$  and  $T^{n_i}x \rightarrow x, T^{m_i}x \rightarrow x, T^{n_i+m_i}x \rightarrow x$ .

Similarly we can define (topologically) intersective of order  $d$  and intersective of order  $d$  (see Appendix A.3). We have

**PROPOSITION 7.4.2.** Let  $d \in \mathbb{N}$  and  $P \subset \mathbb{Z}$ . The following statements are equivalent:

- (1)  $P$  is a Birkhoff recurrence set of order  $d$ .
- (2) Whenever  $(X, T)$  is a minimal t.d.s. and  $U \subset X$  a nonempty open set, there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  such that

$$U \cap \left( \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U \right) \neq \emptyset.$$

- (3)  $P$  is (topologically) intersective of order  $d$ .

**PROOF.** (1)  $\Leftrightarrow$  (2) follows from the proof of Proposition 7.2.3. See Appendix A.3 for the proof (1)  $\Leftrightarrow$  (3).  $\square$

**REMARK 7.4.3.** From the above proof, one can see that for a minimal t.d.s. the set of recurrent point in the Definition 7.4.1 is residual.

**7.4.1.2. Some properties of Birkhoff sequences of order  $d$ .** The family of all Birkhoff recurrence sets of order  $d$  is denoted by  $\mathcal{F}_{B_d}$ . We have

$$\mathcal{F}_{B_1} \supset \mathcal{F}_{B_2} \supset \dots \supset \mathcal{F}_{B_d} \supset \dots \supset \mathcal{F}_{B_\infty} =: \bigcap_{d=1}^{\infty} \mathcal{F}_{B_d}.$$

We will show later (after Proposition 7.4.10) that

**PROPOSITION 7.4.4.**  $\mathcal{F}_{B_\infty} = \mathcal{F}_{fip}$ .

**7.4.2. Birkhoff recurrence sets and  $\mathbf{RP}^{[d]}$ .** We have the following theorem

**THEOREM 7.4.5.** Let  $(X, T)$  be a minimal t.d.s. Then for any  $d \in \mathbb{N} \cup \{\infty\}$ ,  $(x, y) \in \mathbf{RP}^{[d]}$  if and only if  $N(x, U) \in \mathcal{F}_{B_d}$  for each neighborhood  $U$  of  $y$ .

**PROOF.** We first show the case when  $d \in \mathbb{N}$ .

( $\Leftarrow$ ) Let  $d \in \mathbb{N}$  and assume  $N(x, U) \in \mathcal{F}_{B_d}$ . Then there are  $FS(\{n_i\}_{i=1}^d) \subset N(x, U)$  such that  $U \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U \neq \emptyset$ . This means that there is  $y' \in U$  such that  $T^n y' \in U$  for any  $n \in FS(\{n_i\}_{i=1}^d)$ . Since  $T^n x \in U$  for any  $n \in FS(\{n_i\}_{i=1}^d)$ , we conclude that  $(x, y) \in \mathbf{RP}^{[d]}$  by the definition.

( $\Rightarrow$ ) Assume that  $(x, y) \in \mathbf{RP}^{[d]}$  and  $U$  is a neighborhood of  $y$ . Let  $(Z, R)$  be a minimal t.d.s.,  $V$  be a non-empty open subset of  $Z$  and  $\Lambda \subset X \times Z$  be a minimal subsystem. Let  $\pi : \Lambda \rightarrow X$  be the projection. Since  $(x, y) \in \mathbf{RP}^{[d]}$  there are  $z_1, z_2 \in Z$  such that  $((x, z_1), (y, z_2)) \in \mathbf{RP}^{[d]}(\Lambda, T \times R)$  by Theorem 7.1.3. Let  $m \in \mathbb{N}$  such that  $R^{-m}V$  be a neighborhood of  $z_2$ . Then  $U \times R^{-m}V$  is a neighborhood of  $(y, z_2)$ . By Theorem 7.1.3, there are  $n_1, \dots, n_{d+1}$  such that

$$N((x, z_1), U \times R^{-m}V) \supset FS(\{n_i\}_{i=1}^{d+1}).$$

This implies that  $\bigcap_{n \in FS(\{n_i\}_{i=1}^{d+1})} R^{-n-m}V \neq \emptyset$ . Thus,  $V \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} R^{-n}V \neq \emptyset$ , i.e.  $N(x, U) \in \mathcal{F}_{B_d}$ .

The case  $d = \infty$  is followed from the result for  $d \in \mathbb{N}$  and the definitions.  $\square$

### 7.4.3. Cubic version of multiple Poincaré recurrence sets.

7.4.3.1. *Poincaré recurrence sets.* Now we give the cubic version of multiple Poincaré recurrence sets.

DEFINITION 7.4.6. For  $d \in \mathbb{N}$ , a subset  $F$  of  $\mathbb{Z}$  is a *Poincaré recurrence set of order  $d$*  if for each  $(X, \mathcal{B}, \mu, T)$  and  $A \in \mathcal{B}$  with  $\mu(A) > 0$  there are  $n_1, \dots, n_d \in \mathbb{Z}$  such that  $FS(\{n_i\}_{i=1}^d) \subset F$  and

$$\mu(A \cap \left( \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A \right)) > 0.$$

A subset  $F$  of  $\mathbb{Z}$  is a *Poincaré recurrence set of order  $\infty$*  if it is a Poincaré recurrence set of order  $d$  for any  $d \geq 1$ .

REMARK 7.4.7. We remark that  $F$  is a Poincaré recurrence set of order 1 if and only if it is a Poincaré sequence. Moreover, a Poincaré recurrence set of order 1 does not imply that it is a Poincaré recurrence set of order 2. For example,  $\{n^k : n \in \mathbb{N}\}$  ( $k \geq 3$ ) is a Poincaré sequence [21], it is not a Poincaré recurrence set of order 2 by the famous Fermat Last Theorem.

7.4.3.2. *Some properties of Poincaré recurrence sets of order  $d$ .* Let for  $d \in \mathbb{N} \cup \{\infty\}$ ,  $\mathcal{F}_{P_d}$  be the family consisting of all Poincaré recurrence sets of order  $d$ . Thus

$$\mathcal{F}_{P_1} = \mathcal{F}_{P_{oi}} \supset \mathcal{F}_{P_2} \supset \dots \supset \mathcal{F}_{P_d} \supset \dots \supset \mathcal{F}_{P_\infty} =: \bigcap_{d=1}^{\infty} \mathcal{F}_{P_d}.$$

We want to show that  $\mathcal{F}_{P_\infty} = \mathcal{F}_{fip}$ . It is clear that  $\mathcal{F}_{P_\infty} \subset \mathcal{F}_{fip}$ . To show  $\mathcal{F}_{P_d} \supset \mathcal{F}_{fip}$ , we need the following proposition, for a proof see [23] or [40].

PROPOSITION 7.4.8. Let  $(X, \mathcal{B}, \mu)$  be a probability space, and  $\{E_i\}_{i=1}^\infty$  be a sequence of measurable sets with  $\mu(E_i) \geq a > 0$  for some constant  $a$  and any  $i \in \mathbb{N}$ . Then for any  $k \geq 1$  and  $\epsilon > 0$  there is  $N = N(a, k, \epsilon)$  such that for any tuple  $\{s_1 < s_2 < \dots < s_n\}$  with  $n \geq N$  there exist  $1 \leq t_1 < t_2 < \dots < t_k \leq n$  with

$$(7.3) \quad \mu(E_{st_1} \cap E_{st_2} \cap \dots \cap E_{st_k}) \geq a^k - \epsilon.$$



REMARK 7.4.9. To prove Proposition 7.4.10, one needs to use Proposition 7.4.8 repeatedly. To avoid explaining the same idea frequently, we illustrate how we will use Proposition 7.4.8 in the proof of Proposition 7.4.10 first.

For each  $j \in \mathbb{N}$ , let  $\{k_i^j\}_{i=1}^\infty$  be a sequences in  $\mathbb{Z}$ . Assume  $(X, \mathcal{B}, \mu, T)$  is a measure preserving system and  $A \in \mathcal{B}$  with  $\mu(A) > 0$ . Let  $A_1 = A$ ,  $a_1 = \mu(A_1)$ , and  $a_{j+1} = \frac{1}{2}a_j^2$  for all  $j \geq 1$ . We will show that there are a decreasing sequence  $\{A_j\}_j$  of measurable sets and a sequence  $\{N_j\} \subset \mathbb{N}$  such that for each  $j$ ,  $\mu(A_j) \geq \frac{1}{2}a_{j-1}^2 = a_j > 0$ , and for  $n \geq N_j$  and any tuple  $\{s(1) < s(2) < \dots < s(n)\}$  there exist  $1 \leq t(1, j) < t(2, j) \leq n$  with  $\mu(T^{-k_{s(t(1,j))}^j} A_j \cap T^{-k_{s(t(2,j))}^j} A_j) \geq \frac{1}{2}a_j^2 = a_{j+1}$ .

Set  $E_i^1 = T^{-k_i^1} A$ ,  $i \in \mathbb{N}$ . Let  $N_1 = N(a_1, 2, \frac{1}{2}a_1^2)$  be as in Proposition 7.4.8. Then for  $n \geq N_1$  and any tuple  $\{s(1) < \dots < s(n)\}$  there exist  $1 \leq t(1, 1) < t(2, 1) \leq n$  with  $\mu(E_{s(t(1,1))}^1 \cap E_{s(t(2,1))}^1) \geq \frac{1}{2}a_1^2 = a_2$ .

Fix  $t(1, 1) < t(2, 1)$  for a given tuple  $\{s(1) < \dots < s(n)\}$ . Now let  $A_2 = A_1 \cap T^{-k_{s(t(2,1))}^1 + k_{s(t(1,1))}^1} A_1$ . Then  $\mu(A_2) = \mu(E_{s(t(1,1))}^1 \cap E_{s(t(2,1))}^1) \geq \frac{1}{2}a_1^2 = a_2$ . Let  $E_i^2 = T^{-k_i^2} A_2$ ,  $i \in \mathbb{N}$  and  $N_2 = N(a_2, 2, \frac{1}{2}a_2^2)$  be as in Proposition 7.4.8. Thus for  $n \geq N_2$  and any tuple  $\{s(1) < \dots < s(n)\}$  there exist  $1 \leq t(1, 2) < t(2, 2) \leq n$  with  $\mu(E_{s(t(1,2))}^2 \cap E_{s(t(2,2))}^2) \geq \frac{1}{2}a_2^2 = a_3$ .

Inductively, assume that  $\{E_i^j = T^{-k_i^j} A_j\}_{i=1}^\infty$ ,  $A_j$ ,  $a_j$ ,  $N_j$  are defined such that for  $n \geq N_j$  and any tuple  $\{s(1) < \dots < s(n)\}$  there exist  $1 \leq t(1, j) < t(2, j) \leq n$  with  $\mu(E_{s(t(1,j))}^j \cap E_{s(t(2,j))}^j) \geq \frac{1}{2}a_j^2 = a_{j+1}$ .

Fix  $t(1, j) < t(2, j)$  for a given tuple  $\{s(1) < \dots < s(n)\}$ . Let  $A_{j+1} = A_j \cap T^{-k_{s(t(2,j))}^j + k_{s(t(1,j))}^j} A_j$ . Then

$$\mu(A_{j+1}) = \mu(E_{s(t(1,j))}^j \cap E_{s(t(2,j))}^j) \geq \frac{1}{2}a_j^2 = a_{j+1}.$$

Let  $E_i^{j+1} = T^{-k_i^{j+1}} A_{j+1}$ ,  $i \in \mathbb{N}$ , and  $N_{j+1} = N(a_{j+1}, 2, \frac{1}{2}a_{j+1}^2)$  be as in Proposition 7.4.8. Then for  $n \geq N_{j+1}$  and any tuple  $\{s(1) < \dots < s(n)\}$  there exist  $1 \leq t(1, j+1) < t(2, j+1) \leq n$  with  $\mu(E_{s(t(1,j+1))}^{j+1} \cap E_{s(t(2,j+1))}^{j+1}) \geq \frac{1}{2}a_{j+1}^2 = a_{j+2}$ .

Note that the choices of  $\{N_i\}$  is independent of  $\{k_i^j\}_{i=1}^\infty$ .  $\square$

Now we are ready to show

PROPOSITION 7.4.10. The following statements hold.

- (1) For each  $d \in \mathbb{N}$ ,  $\mathcal{F}_{fip} \subset \mathcal{F}_{P_d}$ , which implies that  $\mathcal{F}_{P_\infty} = \mathcal{F}_{fip}$ .
- (2)  $\mathcal{F}_{SG_d} \subset \mathcal{F}_{P_d}$  for each  $d \in \mathbb{N} \cup \{\infty\}$ . Moreover one has  $\mathcal{F}_{fSG_d} \subset \mathcal{F}_{P_d}$ .

PROOF. (1) Let  $F \in \mathcal{F}_{fip}$ . Fix  $d \in \mathbb{N}$ . Now we show  $F \in \mathcal{F}_{P_d}$ . For this purpose, assume that  $(X, \mathcal{B}, \mu, T)$  is a measure preserving system and  $A \in \mathcal{B}$  with  $\mu(A) > 0$ . Since  $F \in \mathcal{F}_{fip}$ , there are  $p_1, p_2, \dots, p_{\ell_d} \in \mathbb{Z}$  with  $\ell_d = \sum_{i=1}^d N_i$  such that  $F \supset FS(\{p_i\}_{i=1}^{\ell_d})$ , where  $N_i$  are chosen as in Remark 7.4.9 for  $(X, \mathcal{B}, \mu, T)$  and  $A$ .

Let  $A_1 = A$ ,  $a_1 = \mu(A_1)$ , and  $a_{j+1} = \frac{1}{2}a_j^2$  for all  $j \geq 1$ . For  $p_1, p_1 + p_2, \dots, p_1 + \dots + p_{N_1}$  by the argument in Remark 7.4.9 (by setting  $\{k_i^1\} = \{p_1, p_1 + p_2, \dots\}$  and  $(s(1), \dots, s(N_1)) = (1, \dots, N_1)$ ) there is  $q_1 = p_{i_1^1} + \dots + p_{i_2^1}$  such that  $\mu(A_1 \cap$

$T^{-q_1}A_1) \geq \frac{1}{2}a_1^2 = a_2$ , where  $1 \leq i_1^1 < i_2^1 \leq N_1$ . Let  $A_2 = A_1 \cap T^{-q_1}A_1$ . For  $p_{N_1+1}, p_{N_1+1} + p_{N_1+2}, \dots, p_{N_1+1} + \dots + p_{N_1+N_2}$ , there is  $q_2 = p_{i_1^2} + \dots + p_{i_2^2}$  such that  $\mu(A_2 \cap T^{-q_2}A_2) \geq \frac{1}{2}a_2^2 = a_3$ , where  $N_1 + 1 \leq i_1^2 < i_2^2 \leq N_1 + N_2$ . Note that  $q_1, q_2, q_1 + q_2 \in F$ .

Inductively we obtain

$$N_1 + \dots + N_j + 1 \leq i_1^{j+1} < i_2^{j+1} \leq N_1 + \dots + N_{j+1}, \quad 0 \leq j \leq d-1.$$

$q_1, \dots, q_d$  and  $A_1, \dots, A_q$  with  $q_j = \sum_{i=i_1^j}^{i_2^j} p_i$  and  $A_j = A_{j-1} \cap T^{-q_{j-1}}A_{j-1}$  such that  $\mu(A_j) \geq a_j$  and  $\mu(A_j \cap T^{-q_j}A_j) \geq \frac{1}{2}a_j^2 = a_{j+1}$ . Thus

$$\mu(A \cap \bigcap_{n \in FS(\{q_i\}_{i=1}^d)} T^{-n}A) \geq \frac{1}{2}a_d^2 > 0,$$

and it is clear that  $F \supset FS(\{q_i\}_{i=1}^d)$ . This implies that  $F \in \mathcal{F}_{P_d}$ .

Thus  $\mathcal{F}_{P_\infty} \supset \mathcal{F}_{fip}$ . Since it is clear that  $\mathcal{F}_{P_\infty} \subset \mathcal{F}_{fip}$ , we are done.

(2) Since each  $SG_1$ -set is a  $\Delta$ -set, it is a Poincaré recurrence set (this is easy to be checked by Poincaré recurrence Theorem [20]). We first show the case when  $d = 2$  which will illustrate the general idea. Then we give the proof for the general case.

Let  $F \in SG_2$ . Then there is  $P = \{P_i\}_{i=1}^\infty \subset \mathbb{Z}$  with  $F = SG_2(P)$ . Let  $(X, \mathcal{B}, \mu, T)$  be a measure preserving system and  $A \in \mathcal{B}$  with  $\mu(A) > 0$ . Set  $A_1 = A$  and  $a_1 = \mu(A_1)$ .

Let

$$q_1 = \sum_{i=1}^{N_2} P_{2i-1}, q_2 = \sum_{i=N_2+1}^{2N_2} P_{2i-1}, \dots, \text{ and } q_{N_1} = \sum_{i=(N_1-1)N_2+1}^{N_1N_2} P_{2i-1},$$

where  $N_1 = N(a_1, 2, \frac{1}{2}a_1^2)$  and  $N_2 = N(a_2, 2, \frac{1}{2}a_2^2)$  are chosen as in Remark 7.4.9 for  $(X, \mathcal{B}, \mu, T)$  and  $A$ . Consider the sequence  $q_1, q_1 + q_2, \dots, q_1 + q_2 + \dots + q_{N_1}$ . Then as in Remark 7.4.9 there are  $1 \leq i_1 \leq j_1 \leq N_1$  such that  $\mu(A_2) \geq \frac{1}{2}\mu(A)^2$ , where  $A_2 = A_1 \cap T^{-n_1}A_1$  and  $n_1 = \sum_{i=i_1}^{j_1} q_i$ . Note that

$$n_1 = P_{2(i_1-1)N_2+1} + P_{2(i_1-1)N_2+3} + \dots + P_{2j_1N_2-1}.$$

Now consider the sequence

$$P_{2(i_1-1)N_2}, P_{2(i_1-1)N_2} + P_{2(i_1-1)N_2+2}, \dots, P_{2(i_1-1)N_2} + P_{2(i_1-1)N_2+2} + \dots + P_{2i_1N_2}.$$

It has  $N_2 + 1$  terms. So as in Remark 7.4.9 there are  $1 \leq i_2 \leq j_2 \leq N_2$  such that  $\mu(A_2 \cap T^{-n_2}A_2) \geq \frac{1}{2}a_2^2$ , where  $n_2 = \sum_{i=(i_1-1)N_2+i_2}^{(i_1-1)N_2+j_2} P_{2i}$ . Note that  $n_1, n_2, n_1 + n_2 \in F$  by the definition of  $SG_2(P)$ . It is easy to verify that

$$\mu(A \cap T^{-n_1}A \cap T^{-n_2}A \cap T^{-n_1-n_2}A) \geq \frac{1}{2}a_2^2 > 0.$$

Hence  $F \in \mathcal{F}_{P_2}$ .

Now we show the general case. Assume that  $d \geq 3$  and let  $F \in SG_d$ . We show that  $F \in \mathcal{F}_{P_d}$ .

Since  $F \in SG_d$ , there is  $P = \{P_i\}_{i=1}^\infty \subset \mathbb{Z}$  with  $F = SG_d(P)$ . Let  $(X, \mathcal{B}, \mu, T)$  be a measure preserving system and  $A \in \mathcal{B}$  with  $\mu(A) > 0$ . Set  $A_1 = A$ . Let  $N_1, \dots, N_d$  be the numbers as defined in Remark 7.4.9 for  $(X, \mathcal{B}, \mu, T)$ ,  $A$  and let  $M_i = \prod_{j=i}^d N_j$  for  $1 \leq i \leq d$ .

Let

$$q_1^1 = \sum_{i=1}^{M_2} P_{di-(d-1)}, \quad q_2^1 = \sum_{i=M_2+1}^{2M_2} P_{di-(d-1)}, \quad \dots, \quad q_{N_1}^1 = \sum_{i=(N_1-1)M_2+1}^{M_1} P_{di-(d-1)}.$$

Consider the sequence  $q_1^1, q_1^1 + q_2^1, \dots, q_1^1 + q_2^1 + \dots + q_{N_1}^1$ . Then as in Remark 7.4.9 there are  $1 \leq i_1 \leq j_1 \leq N_1$  such that  $\mu(A_2) \geq \frac{1}{2}a_1^2$ , where  $A_2 = A_1 \cap T^{-n_1}A_1$  and  $n_1 = \sum_{i=i_1}^{j_1} q_i^1$ .

Let  $m_1 = (i_1 - 1)M_2$ . Note that there is  $t_1 \geq M_2 - 1$  such that

$$n_1 = \sum_{i=i_1}^{j_1} q_i^1 = P_{dm_1+1} + P_{dm_1+d+1} + \dots + P_{dm_1+t_1d+1}.$$

Now consider

$$q_1^2 = \sum_{i=m_1+1}^{m_1+M_3} P_{di-(d-2)}, \quad q_2^2 = \sum_{i=m_1+M_3+1}^{m_1+2M_3} P_{di-(d-2)}, \quad \dots, \quad q_{N_2}^2 = \sum_{i=m_1+(N_2-1)M_3+1}^{m_1+M_2} P_{di-(d-2)}.$$

Now consider  $q_1^2, q_1^2 + q_2^2, \dots, q_1^2 + q_2^2 + \dots + q_{N_2}^2$ . It has  $N_2$  terms. So as in Remark 7.4.9 there are  $1 \leq i_2 \leq j_2 \leq N_2$  such that  $\mu(A_3) \geq \frac{1}{2}a_2^2$ , where  $A_3 = A_2 \cap T^{-n_2}A_2$  and  $n_2 = \sum_{i=i_2}^{j_2} q_i^2$ . Let  $m_2 = m_1 + (i_2 - 1)M_3$ . Note that  $n_1, n_2, n_1 + n_2 \in F$  and there is  $t_2 \geq M_3 - 1$  such that

$$n_2 = \sum_{i=i_2}^{j_2} q_i^2 = P_{dm_2+2} + P_{dm_2+d+2} + \dots + P_{dm_2+t_2d+2}.$$

Note that  $n_2$  has at least  $M_3$  terms.

Inductively for  $1 \leq k \leq d-1$  we have  $1 \leq i_k \leq j_k \leq N_k$  and

$$n_k = \sum_{i=i_k}^{j_k} q_i^k = P_{dm_k+k} + P_{dm_k+d+k} + \dots + P_{dm_k+t_kd+k},$$

where  $t_k \geq M_{k+1} - 1$ . Also we have  $A_k = A_{k-1} \cap T^{-n_{k-1}}A_{k-1}$  with  $\mu(A_k) \geq \frac{1}{2}a_{k-1}^2$ , and  $FS(\{n_j\}_{j=1}^k) \subset F$ .

Especially, when  $k = d$ , we get  $1 \leq i_d \leq j_d \leq N_d$  and  $n_d = \sum_{i=i_d}^{j_d} P_{di}$ . By the definition of  $SG_d$  we get that  $FS(\{n_i\}_{i=1}^d) \subset F$ . From the definition of  $A_j, j = 1, 2, \dots, d$ , one has

$$\mu(A \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A) \geq \frac{1}{2}a_d^2 > 0,$$

which implies that  $F \in \mathcal{F}_{P_d}$ . The proof is completed.  $\square$

7.4.3.3. *Proof of Proposition 7.4.4:* It is clear that  $\mathcal{F}_{B_\infty} \subset \mathcal{F}_{fip}$ . Since  $\mathcal{F}_{fip} \subset \mathcal{F}_{P_\infty} \subset \mathcal{F}_{B_\infty}$  (by Proposition 7.4.10 and the obvious fact that  $\mathcal{F}_{P_d} \subset \mathcal{F}_{B_d}$ ) we have  $\mathcal{F}_{B_\infty} = \mathcal{F}_{fip}$ .

#### 7.4.4. Poincaré recurrence sets and $\mathbf{RP}^{[d]}$ .

THEOREM 7.4.11. Let  $(X, T)$  be a minimal t.d.s. Then for each  $d \in \mathbb{N} \cup \{\infty\}$ ,  $(x, y) \in \mathbf{RP}^{[d]}$  if and only if  $N(x, U) \in \mathcal{F}_{P_d}$  for any neighborhood  $U$  of  $y$ .

PROOF. We first show the case when  $d \in \mathbb{N}$ . ( $\Leftarrow$ ) Since  $\mathcal{F}_{P_d} \subset \mathcal{F}_{B_d}$ , it follows from Theorem 7.4.5.

( $\Rightarrow$ ) Assume that  $(x, y) \in \mathbf{RP}^{[d]}$  and  $U$  is a neighborhood of  $y$ . By Theorem 7.3.2,  $N(x, U) \in \mathcal{F}_{SG_d}$ . Then by Proposition 7.4.10 we have  $N(x, U) \in \mathcal{F}_{P_d}$ .

The case  $d = \infty$  follows from the case  $d \in \mathbb{N}$  and definitions.  $\square$

### 7.5. Conclusion

Now we sum up the results of previous three sections. Note that  $\mathcal{F}_{Bir_\infty}$  and  $\mathcal{F}_{Poi_\infty}$  can be defined naturally. Since  $\mathcal{F}_{1,0} \subset \mathcal{F}_{2,0} \subset \dots$  we define  $\mathcal{F}_{\infty,0} =: \bigcup_{d=1}^{\infty} \mathcal{F}_{d,0}$ . Another way to do this is that one follows the idea in [13] to define  $\infty$ -step nilsystems and view  $\mathcal{F}_{\infty,0}$  as the family generated by all  $\text{Nil}_\infty$  Bohr $_0$ -sets. It is easy to check that Theorem 7.2.7 holds for  $d = \infty$ .

Thus we have

THEOREM 7.5.1. Let  $(X, T)$  be a minimal t.d.s. and  $x, y \in X$ . Then the following statements are equivalent for  $d \in \mathbb{N} \cup \{\infty\}$ :

- (1)  $(x, y) \in \mathbf{RP}^{[d]}$ .
- (2)  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ .
- (3)  $N(x, U) \in \mathcal{F}_{Poi_d}$  for each neighborhood  $U$  of  $y$ .
- (4)  $N(x, U) \in \mathcal{F}_{Bir_d}$  for each neighborhood  $U$  of  $y$ .
- (5)  $N(x, U) \in \mathcal{F}_{SG_d}$  for each neighborhood  $U$  of  $y$ .
- (6)  $N(x, U) \in \mathcal{F}_{fSG_d}$  for each neighborhood  $U$  of  $y$ .
- (7)  $N(x, U) \in \mathcal{F}_{B_d}$  for each neighborhood  $U$  of  $y$ .
- (8)  $N(x, U) \in \mathcal{F}_{P_d}$  for each neighborhood  $U$  of  $y$ .

## CHAPTER 8

### *d*-step almost automorphy and recurrence sets

In the previous chapter we obtain some characterizations of regionally proximal relation of order  $d$ . In the present section we study  $d$ -step almost automorphy.

#### 8.1. Definition of $d$ -step almost automorphy

8.1.0.1. First we recall the notion of  $d$ -step almost automorphic systems and give its structure theorem.

**DEFINITION 8.1.1.** Let  $(X, T)$  be a t.d.s. and  $d \in \mathbb{N} \cup \{\infty\}$ . A point  $x \in X$  is called a  *$d$ -step almost automorphic point* (or  $d$ -step AA point for short) if  $\mathbf{RP}^{[d]}(Y)[x] = \{x\}$ , where  $Y = \overline{\{T^n x : n \in \mathbb{Z}\}}$  and  $\mathbf{RP}^{[d]}(Y)[x] = \{y \in Y : (x, y) \in \mathbf{RP}^{[d]}(Y)\}$ .

A minimal t.d.s.  $(X, T)$  is called  *$d$ -step almost automorphic* ( $d$ -step AA for short) if it has a  $d$ -step almost automorphic point.

**REMARK 8.1.2.** Since

$$\mathbf{RP}^{[\infty]} \subset \dots \subset \mathbf{RP}^{[d]} \subset \mathbf{RP}^{[d-1]} \subset \dots \subset \mathbf{RP}^{[1]},$$

we have

$$\text{AA} = 1\text{-step AA} \Rightarrow \dots \Rightarrow (d-1)\text{-step AA} \Rightarrow d\text{-step AA} \Rightarrow \dots \Rightarrow \infty\text{-step AA}.$$

8.1.0.2. The following theorem follows from Theorem 7.1.3.

**THEOREM 8.1.3** (Structure of  $d$ -step almost automorphic systems). Let  $(X, T)$  be a minimal t.d.s. Then  $(X, T)$  is a  $d$ -step almost automorphic system for some  $d \in \mathbb{N} \cup \{\infty\}$  if and only if it is an almost one-to-one extension of its maximal  $d$ -step nilfactor  $(X_d, T)$ .

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \downarrow & & \downarrow \pi \\ X_d & \xrightarrow{T} & X_d \end{array}$$

**8.1.1. 1-step almost automorphy.** First we recall some classical results about almost automorphy.

Let  $(X, T)$  be a minimal t.d.s.. In [52] it is proved that  $(x, y) \in \mathbf{RP}^{[1]}$  if and only if for each neighborhood  $U$  of  $y$ ,  $N(x, U)$  contains some  $\Delta$ -set, see also Theorem 7.3.2. Similarly, we have that for a minimal system  $(X, T)$ ,  $(x, y) \in \mathbf{RP}^{[1]}$  if and only if for each neighborhood  $U$  of  $y$ ,  $N(x, U) \in \mathcal{F}_{Poi}$  [40], see also Theorem 7.5.1.

Using these theorems and the facts that  $\mathcal{F}_{Poi}$  and  $\mathcal{F}_{Bir}$  have the Ramsey property, one has

**THEOREM 8.1.4.** Let  $(X, T)$  be a minimal t.d.s. and  $x \in X$ . Then the following statements are equivalent:

- (1)  $x$  is AA.
- (2)  $N(x, V) \in \mathcal{F}_{Poi}^*$  for each neighborhood  $V$  of  $x$ .
- (3)  $N(x, V) \in \mathcal{F}_{Bir}^*$  for each neighborhood  $V$  of  $x$ .
- (4)  $N(x, V) \in \Delta^*$  for each neighborhood  $V$  of  $x$ . [51, 21]

We will not give the proof of this theorem since it is a special case of Theorem 8.2.1.

**8.1.2.  $\infty$ -step almost automorphy.** In this subsection we give one characterization for  $\infty$ -step AA. Following from Theorem 7.1.3, one has

**PROPOSITION 8.1.5.** Let  $(X, T)$  be a minimal t.d.s. and  $d \geq 1$ . Then

- (1)  $(x, y) \in \mathbf{RP}^{[d]}$  if and only if  $N(x, U)$  contains a finite IP-set of length  $d + 1$  for any neighborhood  $U$  of  $y$ , and thus
- (2)  $(x, y) \in \mathbf{RP}^{[\infty]}$  if and only if  $N(x, U) \in \mathcal{F}_{fip}$  for any neighborhood  $U$  of  $y$ .

To show the next theorem we need the following lemma which should be known, see for example Huang, Li and Ye [39].

**LEMMA 8.1.6.**  $\mathcal{F}_{fip}$  has the Ramsey property.

We have the following

**THEOREM 8.1.7.** Let  $(X, T)$  be a minimal t.d.s. Then  $(X, T)$  is  $\infty$ -step AA if and only if there is  $x \in X$  such that  $N(x, V) \in \mathcal{F}_{fip}^*$  for each neighborhood  $V$  of  $x$ .

**PROOF.** Assume that there is  $x \in X$  such that  $N(x, V) \in \mathcal{F}_{fip}^*$  for each neighborhood  $V$  of  $x$ . If there is  $y \in X$  such that  $(x, y) \in \mathbf{RP}^{[\infty]}$ , then by Proposition 8.1.5 for any neighborhood  $U$  of  $y$ ,  $N(x, U) \in \mathcal{F}_{fip}$ . This implies that  $x = y$ , i.e.  $(X, T)$  is  $\infty$ -step AA.

Now assume that  $(X, T)$  is  $\infty$ -step AA, i.e. there is  $x \in X$  such that  $\mathbf{RP}^{[\infty]}[x] = \{x\}$ . If for some neighborhood  $V$  of  $x$ ,  $N(x, V) \notin \mathcal{F}_{fip}^*$ , then  $N(x, V^c)$  contains finite IP-sets of arbitrarily long lengths.

Let  $U_1 = V^c$ . Covering  $U_1$  by finitely many closed balls  $U_1^1, \dots, U_1^{i_1}$  of diam  $\leq 1$ . Then there is  $j_1$  such that  $N(x, U_1^{j_1})$  contains finite IP-sets of arbitrarily long lengths. Let  $U_2 = U_1^{j_1}$ . Covering  $U_2$  by finitely many closed balls  $U_2^1, \dots, U_2^{i_2}$  of diam  $\leq \frac{1}{2}$ . Then there is  $j_2$  such that  $N(x, U_2^{j_2})$  contains finite IP-sets of arbitrarily long lengths. Let  $U_3 = U_2^{j_2}$ . Inductively, there are a sequence of closed balls  $U_n$  with diam  $\leq \frac{1}{n}$  such that  $N(x, U_n)$  contains finite IP-sets of arbitrarily long lengths. Let  $\{y\} = \bigcap U_n$ . It is clear that  $(x, y) \in \mathbf{RP}^{[\infty]}$  with  $y \neq x$ , a contradiction. Thus  $N(x, V) \in \mathcal{F}_{fip}^*$  for each neighborhood  $V$  of  $x$ .  $\square$

### 8.2. Characterization of $d$ -step almost automorphy

Now we use the results built in previous sections to get the following characterization for  $d$ -step AA via recurrence sets.

**THEOREM 8.2.1.** Let  $(X, T)$  be a minimal t.d.s.,  $x \in X$  and  $d \in \mathbb{N} \cup \{\infty\}$ . Then the following statements are equivalent:

- (1)  $x$  is a  $d$ -step AA point.
- (2)  $N(x, V) \in \mathcal{F}_{d,0}$  for each neighborhood  $V$  of  $x$ .
- (3)  $N(x, V) \in \mathcal{F}_{Poi_d}^*$  for each neighborhood  $V$  of  $x$ .
- (4)  $N(x, V) \in \mathcal{F}_{Bir_d}^*$  for each neighborhood  $V$  of  $x$ .

**PROOF.** Roughly speaking this theorem follows from Theorem 7.5.1, the fact  $\mathcal{F}_{d,0}^*$ ,  $\mathcal{F}_{Poi_d}$  and  $\mathcal{F}_{Bir_d}$  have the Ramsey property, and the idea of the proof of Theorem 8.1.7. We show that (1)  $\Leftrightarrow$  (2), and the rest is similar.

(1)  $\Rightarrow$  (2): Let  $x$  be a  $d$ -step AA point. If (2) does not hold, then there is some neighborhood  $V$  of  $x$  such that  $N(x, V) \notin \mathcal{F}_{d,0}$ . Then  $N(x, V^c) = \mathbb{Z} \setminus N(x, V) \in \mathcal{F}_{d,0}^*$ . Since  $\mathcal{F}_{d,0}^*$  has the Ramsey property, similar to the proof of Theorem 8.1.7 one can find some  $y \in V^c$  such that  $N(x, U) \in \mathcal{F}_{d,0}^*$  for every neighborhood  $U$  of  $y$ . By Theorem 7.5.1,  $y \in \mathbf{RP}^{[d]}[x]$ . Since  $y \neq x$ , this contradicts the fact  $x$  being  $d$ -step AA.

(2)  $\Rightarrow$  (1): If  $x$  is not  $d$ -step AA, then there is some  $y \in \mathbf{RP}^{[d]}[x]$  with  $x \neq y$ . Let  $U_x$  and  $U_y$  be neighborhoods of  $x$  and  $y$  with  $U_x \cap U_y = \emptyset$ . By (2)  $N(x, U_x) \in \mathcal{F}_{d,0}$ . By Theorem 7.5.1,  $N(x, U_y) \in \mathcal{F}_{d,0}^*$ . Hence  $N(x, U_x) \cap N(x, U_y) \neq \emptyset$ , which contradicts the fact that  $U_x \cap U_y = \emptyset$ .  $\square$





## APPENDIX A

### A.1. The Ramsey properties

Recall that a family  $\mathcal{F}$  has the *Ramsey property* means that if  $A \in \mathcal{F}$  and  $A = \bigcup_{i=1}^n A_i$  then one of  $A_i$  is still in  $\mathcal{F}$ . In this section, we show that  $\mathcal{F}_{SG_2}$  does not have the Ramsey property.

**THEOREM A.1.1.**  $\mathcal{F}_{SG_2}$  does not have the Ramsey property.

**PROOF.** Let  $P = \{p_1, p_2, \dots\}$  be a subsequence of  $\mathbb{N}$  with  $p_{i+1} > 2(p_1 + \dots + p_i)$ . The assumption that  $p_{i+1} > 2(p_1 + \dots + p_i)$  ensures that each element of  $SG_2(P)$  has a unique expression with the form of  $\sum_i p_{j_i}$ .

Now divide the set  $SG_2(P)$  into the following three sets:

$$\begin{aligned} B_1 &= \{p_{2n-1} + \dots + p_{2m-1} : n \leq m \in \mathbb{N}\} = SG_1(\{p_1, p_3, \dots\}), \\ B_2 &= \{p_{2n} + \dots + p_{2m} : n \leq m \in \mathbb{N}\} = SG_1(\{p_2, p_4, \dots\}), \\ B_0 &= SG_2(P) \setminus (B_1 \cup B_2). \end{aligned}$$

We show that  $B_i \notin \mathcal{F}_{SG_2}$  for  $i = 0, 1, 2$ . In fact, we will prove that for each  $i = 0, 1, 2$  there do not exist  $a_1 \leq a_2 \leq a_3$  such that

$$(*) \quad a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_1 + a_3 \in B_i,$$

which obviously implies that  $B_i \notin \mathcal{F}_{SG_2}$  for  $i = 0, 1, 2$ .

(1). First we show  $B_2 \notin \mathcal{F}_{SG_2}$ . The proof  $B_1 \notin \mathcal{F}_{SG_2}$  follows similarly. Assume the contrary, i.e. there exist  $a_1 \leq a_2 \leq a_3$  such that

$$a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_1 + a_3 \in B_2.$$

Let

$$\begin{aligned} a_1 &= p_{2n_1} + \dots + p_{2m_1}, \quad n_1 \leq m_1; \\ a_2 &= p_{2n_2} + \dots + p_{2m_2}, \quad n_2 \leq m_2; \\ a_3 &= p_{2n_3} + \dots + p_{2m_3}, \quad n_3 \leq m_3. \end{aligned}$$

Since  $a_1 \leq a_2 \leq a_3$  and the assumption that  $p_{i+1} > 2(p_1 + \dots + p_i)$ , one has that  $m_1 \leq m_2 \leq m_3$ . Since  $a_1 + a_2, a_2 + a_3 \in B_2$ , one has that  $n_2 = m_1 + 1$  and  $n_3 = m_2 + 1$ . Hence  $n_3 = m_2 + 1 \geq n_2 + 1 = m_1 + 2$ , i.e.  $n_3 > m_1 + 1$ . Thus

$$a_1 + a_3 \notin B_2,$$

a contraction!

(2). Now we show  $B_0 \notin \mathcal{F}_{SG_2}$ . Assume the contrary, i.e. there exist  $a_1 \leq a_2 \leq a_3$  such that

$$a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_1 + a_3 \in B_0.$$

Let

$$a_1 = p_{i_1^1} + p_{i_2^1} + \dots + p_{i_{k_1}^1};$$

$$a_2 = p_{i_1^2} + p_{i_2^2} + \dots + p_{i_{k_2}^2};$$

$$a_3 = p_{i_1^3} + p_{i_2^3} + \dots + p_{i_{k_3}^3},$$

where  $i_1^r < i_2^r < \dots < i_{k_r}^r$ ,  $i_{j+1}^r \leq i_j^r + 2$  for  $1 \leq j \leq k_r - 1$ , and there are both even and odd numbers in  $\{i_1^r, i_2^r, \dots, i_{k_r}^r\}$  ( $r = 1, 2, 3$ ).

Since there are both even and odd numbers in  $\{i_1^r, i_2^r, \dots, i_{k_r}^r\}$  ( $r = 1, 2, 3$ ) and  $i_{j+1}^r \leq i_j^r + 2$  for  $1 \leq j \leq k_r - 1$ , there exist  $1 \leq j_r \leq k_r - 1$  such that  $i_{j_r+1}^r = i_{j_r}^r + 1$ . Since  $a_1 \leq a_2 \leq a_3$  and the assumption that  $p_{i+1} > 2(p_1 + \dots + p_i)$ , one has that  $i_{k_1}^1 \leq i_{k_2}^2 \leq i_{k_3}^3$ . Note that we have

$$i_1^1 < i_2^1 < \dots < i_{j_1}^1 < i_{j_1+1}^1 = i_{j_1}^1 + 1 < \dots < i_{k_1}^1,$$

$$i_1^2 < i_2^2 < \dots < i_{j_2}^2 < i_{j_2+1}^2 = i_{j_2}^2 + 1 < \dots < i_{k_2}^2,$$

$$i_1^3 < i_2^3 < \dots < i_{j_3}^3 < i_{j_3+1}^3 = i_{j_3}^3 + 1 < \dots < i_{k_3}^3.$$

The condition  $a_1 + a_2 \in B_0$  implies that

$$(a) \quad i_{j_1+1}^1 < i_1^2 \leq i_{k_1}^1 + 2; \quad i_{k_1}^1 < i_{j_2}^2.$$

In fact if  $i_1^2 < i_{j_1}^1$ , then the gap  $\{i_{j_1}^1, i_{j_1}^1 + 1\}$  is missing in the term of  $a_2$  and it contradicts the assumption  $a_2 \in SG_2(P) \in \mathcal{F}_{SG_2}$ . The statement  $i_{k_1}^1 < i_{j_2}^2$  follows by the same argument.

Similarly, using the assumptions  $a_2 + a_3 \in B_0$  and  $a_1 + a_3 \in B_0$ , one has

$$(b) \quad i_{j_2+1}^2 < i_1^3 \leq i_{k_2}^2 + 2; \quad i_{k_2}^2 < i_{j_3}^3.$$

and

$$(c) \quad i_{j_1+1}^1 < i_1^3 \leq i_{k_1}^1 + 2; \quad i_{k_1}^1 < i_{j_3}^3.$$

From (a), we have that  $i_{k_1}^1 < i_{j_2}^2$ ; and from (b), we have  $i_{j_2+1}^2 = i_{j_2}^2 + 1 < i_1^3$ . Hence we have  $i_1^3 \geq i_{k_1}^1 + 3$ , which contradicts (c). The proof is completed.  $\square$

## A.2. Compact Hausdorff systems

In this section we discuss compact Hausdorff systems, i.e. the systems with phase space being compact Hausdorff. The reason for this is not generalization for generalization's sake, but rather that we have to deal with non-metrizable systems. For example, we will use (in the proof of Theorem 7.2.5) an important tool named Ellis semigroup which is a subspace of an uncountable product of copies of the phase space and therefore in general not metrizable.

**A.2.1. Compact Hausdorff systems.** In the classical theory of abstract topological dynamics, the basic assumption about the system is that the space is a compact Hausdorff space and the action group is a topological group. In this paper, we mainly consider the compact metrizable system under  $\mathbb{Z}$ -actions, but in some occasions we have to deal with compact Hausdorff spaces which are non-metrizable. Note that each compact Hausdorff space is a uniform space, and one may use the uniform structure replacing the role of a metric, see for example the Appendix of [2].

First we recall a classical equality concerning regionally proximal relation in compact Hausdorff systems. A *compact Hausdorff system* is a pair  $(X, T)$ , where  $X$  is a compact Hausdorff space and  $T : X \rightarrow X$  is a homeomorphism. Let  $(X, T)$  be a compact Hausdorff system and  $\mathcal{U}_X$  be the unique uniform structure of  $X$ . The *regionally proximal relation* on  $X$  is defined by

$$\mathbf{RP} = \bigcap_{\alpha \in \mathcal{U}_X} \overline{\bigcup_{n \in \mathbb{Z}} (T \times T)^{-n} \alpha}$$

**A.2.2. Ellis semigroup.** A beautiful characterization of distality was given by R. Ellis using so-called enveloping semigroup. Given a compact Hausdorff system  $(X, T)$ , its *enveloping semigroup* (or *Ellis semigroup*)  $E(X, T)$  is defined as the closure of the set  $\{T^n : n \in \mathbb{Z}\}$  in  $X^X$  (with its compact, usually non-metrizable, pointwise convergence topology). Ellis showed that a compact Hausdorff system  $(X, T)$  is distal if and only if  $E(X, T)$  is a group if and only if every point in  $(X^2, T \times T)$  is minimal [14].

**A.2.3. Limits of Inverse systems.** Suppose that every  $\lambda$  in a set  $\Lambda$  directed by the relation  $\leq$  corresponds a t.d.s.  $(X_\lambda, T_\lambda)$ , and that for any  $\lambda, \xi \in \Lambda$  satisfying  $\xi \leq \lambda$  a factor map  $\pi_\xi^\lambda : (X_\lambda, T_\lambda) \rightarrow (X_\xi, T_\xi)$  is defined; suppose further that  $\pi_\tau^\xi \pi_\xi^\lambda = \pi_\tau^\lambda$  for all  $\lambda, \xi, \tau \in \Lambda$  with  $\tau \leq \xi \leq \lambda$  and that  $\pi_\lambda^\lambda = \text{id}_X$  for all  $\lambda \in \Lambda$ . In this situation we say that the family  $\{X_\lambda, \pi_\xi^\lambda, \Lambda\} = \{(X_\lambda, T_\lambda), \pi_\xi^\lambda, \Lambda\}$  is an *inverse system of the systems*  $(X_\lambda, T_\lambda)$ ; and the mappings  $\pi_\xi^\lambda$  are called *bonding mappings* of the inverse system.

Let  $\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$  be an inverse system. The *limit of the inverse system*  $\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$  is the set

$$\left\{ (x_\lambda)_\lambda \in \prod_{\lambda \in \Lambda} X_\lambda : \pi_\xi^\lambda(x_\lambda) = x_\xi \text{ for all } \xi \leq \lambda \in \Lambda \right\},$$

and is denoted by  $\varprojlim \{X_\lambda, \pi_\xi^\lambda, \Lambda\}$ . Let  $X = \varprojlim \{X_\lambda, \pi_\xi^\lambda, \Lambda\}$ . For each  $\lambda \in \Lambda$ , let  $\pi_\lambda : X \rightarrow X_\lambda, (x_\sigma)_\sigma \mapsto x_\lambda$  be the projection mapping.

A well known result is the following (see for example [43]):

**LEMMA A.2.1.** Each compact Hausdorff system is the inverse limit of topological dynamical systems.

**A.2.4. The regionally proximal relation of order  $d$  for compact Hausdorff systems.** The definition of the regionally proximal relation of order  $d$  for compact Hausdorff systems is similar to the metric case.

DEFINITION A.2.2. Let  $(X, T)$  be a compact Hausdorff system,  $\mathcal{U}_X$  be the unique uniform structure of  $X$  and let  $d \geq 1$  be an integer. A pair  $(x, y) \in X \times X$  is said to be *regionally proximal of order  $d$*  if for any  $\alpha \in \mathcal{U}_X$ , there exist  $x', y' \in X$  and a vector  $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$  such that  $(x, x') \in \alpha$ ,  $(y, y') \in \alpha$ , and

$$(T^{\mathbf{n} \cdot \epsilon} x', T^{\mathbf{n} \cdot \epsilon} y') \in \alpha \text{ for any } \epsilon \in \{0, 1\}^d, \epsilon \neq (0, \dots, 0),$$

where  $\mathbf{n} \cdot \epsilon = \sum_{i=1}^d \epsilon_i n_i$ . The set of all regionally proximal pairs of order  $d$  is denoted by  $\mathbf{RP}^{[d]}(X)$ , which is called *the regionally proximal relation of order  $d$* .

By Lemma A.2.1, each compact Hausdorff system is the inverse limit of topological dynamical systems. Recall the definition of the product uniformity. Let  $(X_\lambda, \mathcal{U}_\lambda)_{\lambda \in \Lambda}$  be a family of uniform spaces and let  $Z = \prod_{\lambda \in \Lambda} X_\lambda$ . The uniformity on  $Z$  (the product uniformity) is defined as follows. If  $F = \{\lambda_1, \dots, \lambda_m\}$  is a finite subset of the index set  $\Lambda$  and  $\alpha_{\lambda_j} \in \mathcal{U}_{\lambda_j}$  ( $j = 1, \dots, m$ ), let

$$\Phi_{\alpha_{\lambda_1}, \dots, \alpha_{\lambda_m}} = \{(x, y) \in Z \times Z : (x_{\lambda_j}, y_{\lambda_j}) \in \alpha_{\lambda_j}, j = 1, \dots, m\}.$$

The collection of all such sets  $\Phi_{\alpha_{\lambda_1}, \dots, \alpha_{\lambda_m}}$  for all finite subsets  $F$  of  $\Lambda$  is a base for the product uniformity. From this and the definition of the regionally proximal relation of order  $d$ , one has the following result.

PROPOSITION A.2.3. Let  $(X, T)$  be a compact Hausdorff system and  $d \in \mathbb{N}$ . Suppose that  $X = \varprojlim \{X_\lambda, \pi_\xi^\lambda, \Lambda\}$ , where  $(X_\lambda, T_\lambda)_{\lambda \in \Lambda}$  are t.d.s.. Then

$$\mathbf{RP}^{[d]}(X) = \varprojlim \{\mathbf{RP}^{[d]}(X_\lambda), \pi_\xi^\lambda \times \pi_\xi^\lambda, \Lambda\}.$$

Thus combining this proposition with Theorem 7.1.3, one has

THEOREM A.2.4. Let  $(X, T)$  be a minimal compact Hausdorff system and  $d \in \mathbb{N}$ . Then

- (1)  $\mathbf{RP}^{[d]}(X)$  is an equivalence relation, and so is  $\mathbf{RP}^{[\infty]}$ .
- (2) If  $\pi : (X, T) \rightarrow (Y, S)$  is a factor map, then  $(\pi \times \pi)(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$ .
- (3)  $(X/\mathbf{RP}^{[d]}, T)$  is the maximal  $d$ -step nilfactor of  $(X, T)$ .

Note that for a compact Hausdorff system  $(X, T)$  we say that it is a system of order  $d$  for some  $d \in \mathbb{N}$  if it is an inverse limit of  $d$ -step minimal nilsystems.

### A.3. Intersective

It is well known that  $P$  is a Birkhoff recurrence set if and only if  $P \cap (F - F) \neq \emptyset$  for each  $F \in \mathcal{F}_s$ . To give a similar characterization we have

DEFINITION A.3.1. A subset  $P$  is *topologically intersective of order  $d$*  if for each  $F \in \mathcal{F}_s$  there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  and  $a \in F$  with  $a + FS(\{n_i\}_{i=1}^d) \subset F$ , i.e.  $F \cap \bigcap_{m \in FS(\{n_i\}_{i=1}^d)} (F - n) \neq \emptyset$ .

**THEOREM A.3.2.** A subset  $P$  is topologically intersective of order  $d$  if and only if it is a Birkhoff recurrence set of order  $d$ .

**PROOF.** Assume that  $P$  is a Birkhoff recurrence set of order  $d$ . Let  $F \in \mathcal{F}_s$ . Then  $1_F \in \{0, 1\}^{\mathbb{Z}}$ . Let  $(X, T)$  be a minimal subsystem of  $(\overline{\mathcal{O}(1_F, T)}, T)$ , where  $T$  is the shift. Since  $F \in \mathcal{F}_s$ ,  $[1]$  is a non-empty open subset of  $X$ . By the definition there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  such that  $[1] \cap (\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}[1]) \neq \emptyset$ . It implies that there is  $a \in F$  with  $a + FS(\{n_i\}_{i=1}^d) \subset F$  and hence  $P$  is topologically intersective of order  $d$ .

Assume that  $P$  is topologically intersective of order  $d$ . Let  $(X, T)$  be a minimal t.d.s. and  $U$  be an open non-empty subset. Take  $x \in U$ , then  $F = N(x, U) \in \mathcal{F}_s$ . Thus there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  and  $a \in F$  with  $a + FS(\{n_i\}_{i=1}^d) \subset F$ . It follows that  $U \cap (\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U) \neq \emptyset$ .  $\square$

It is well known that  $P$  is a Poincaré recurrence set if and only if  $P \cap (F - F) \neq \emptyset$  for each  $F \in \mathcal{F}_{pubd}$ . To give a similar characterization we have

**DEFINITION A.3.3.** A subset  $P$  is intersective of order  $d$  if for each  $F \in \mathcal{F}_{pubd}$  there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  and  $a \in F$  with  $a + FS(\{n_i\}_{i=1}^d) \subset F$ .

**THEOREM A.3.4.** A subset is intersective of order  $d$  if and only if it is a Poincaré recurrence set of order  $d$ .

**PROOF.** Assume that  $P$  is intersective of order  $d$ . Let  $(X, \mathcal{B}, \mu, T)$  be a measure preserving system and  $A \in \mathcal{B}$  with  $\mu(A) > 0$ . By the Furstenberg corresponding principle, there exists  $F \subset \mathbb{Z}$  such that  $D^*(F) \geq \mu(A)$  and

$$(A.1) \quad \{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} (F - n) \neq \emptyset\} \subset \{\alpha \in \mathcal{F}(\mathbb{Z}) : \mu(\bigcap_{n \in \alpha} T^{-n}A) > 0\},$$

where  $\mathcal{F}(\mathbb{Z})$  denotes the collection of non-empty finite subsets of  $\mathbb{Z}$ . Since  $P$  is intersective of order  $d$ , there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  and  $a \in F$  with  $a + FS(\{n_i\}_{i=1}^d) \subset F$ , i.e.  $F \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} (F - n) \neq \emptyset$ . By (A.1)  $P \in \mathcal{F}_{P_d}$ .

Now assume that  $P \in \mathcal{F}_{P_d}$  and  $F \in \mathcal{F}_{pubd}$ . Then by the Furstenberg corresponding principle, there are a measure preserving system  $(X, \mathcal{B}, \mu, T)$  and  $A \in \mathcal{B}$  such that  $\mu(A) = BD^*(F) > 0$  and

$$(A.2) \quad BD^*(\bigcap_{n \in \alpha} (F - n)) \geq \mu(\bigcap_{n \in \alpha} T^{-n}A)$$

for all  $\alpha \in \mathcal{F}(\mathbb{Z})$ . Since  $P \in \mathcal{F}_{P_d}$ , there are  $n_1, \dots, n_d$  with  $FS(\{n_i\}_{i=1}^d) \subset P$  and  $\mu(A \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A) > 0$ . This implies  $F \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} (F - n) \neq \emptyset$  by (A.2).  $\square$



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